PAUL HALMOS - EXPOSITOR *PAR EXCELLENCE*

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Paul Richard Halmos, one of the best expositors of mathematics - be it with pen on paper or with chalk on blackboard - passed away on October 2, 2006 after a brief period of illness. This article is an attempt to pay homage to him by recalling some of his contributions to mathematics.

Here is what Donald Sarason - arguably the most accomplished Ph.D. student of Halmos - writes about his extraordinary teacher (in [Sel1]):

"Halmos is renowned as an expositor. His writing is something he works hard at, thinks intensely about, and is fiercely proud of. (Witness: "How to write mathematics" (see [Sel2]).) In his papers, he is not content merely to present proofs that are well-organized and clearly expressed; he also suggests the thought processes that went into the construction of his proofs, pointing out the pitfalls he encountered and indicating helpful analogies. His writings clearly reveal his commitment as an educator. In fact, Halmos is instinctively a teacher, a quality discernible in all his mathematical activities, even the most casual ones.

Most of us, when we discover a new mathematical fact, however minor, are usually eager to tell someone about it, to display our cleverness. Halmos behaves differently: he will not tell you his discovery, he will ask you about it, and challenge you to find a proof. If you find a better proof than his, he will be delighted, because then you and he will have taught each other.

To me, Halmos embodies the ideal mixture of researcher and teacher. In him, each role is indistinguishable from the other. Perhaps that is the key to his remarkable influence."

Many of his expository writings (elaborating on his views on diverse topics - writing, lecturing, and doing mathematics - are a 'must read' for every serious student of mathematics. Conveniently, many of them have been collected together in [Sel2].

And here is what one finds in the web pages of the Mathematics Association of America (MAA):

"Professor Halmos was a famed author, editor, teacher, and speaker of distinction. Nearly all of his many books are still in print. His *Finite Dimensional Vector Spaces, Naive Set Theory, Measure Theory, Problems for Mathematicians Young and Old,* and *I Want to be a Mathematician* are classic books that reflect his clarity, conciseness, and color. He edited the American Mathematical Monthly from 1981-1985, and served for many years as one of the editors of the Springer-Verlag series *Undergraduate Texts in Mathematics* and *Graduate Texts in Mathematics.*

Paul was also a great friend of the MAA. Several years ago, Paul and his wife Virginia made a very sizable donation to the MAA for the reconstruction of our

Carriage House in Washington, DC, as a meeting center. That project is just now reaching completion, and already mathematics events are being held there. We at the MAA hope that this wonderful facility will be a fitting tribute to Paul and his mathematical interests."

While Halmos will be the first to acknowledge that there were far more accomplished mathematicians around him, he would at the same time be the last to be apologetic about what he did. There was the famous story of how, as a young and very junior faculty member at the University of Chicago, he would not let himself be bullied by the very senior faculty member Andre Weil on a matter of faculty recruitment. His attitude - which functional analysts everywhere can do well to remember and take strength from - was that while algebaric geometry might be very important, the usefulness of operator theory should not be denied.

Even in his own area of specialisation, there were many mathematicians more powerful than he; but he 'had a nose' for what to ask and which notions to concentrate on. The rest of this article is devoted to trying to justify the assertion of the last line (and describing some of the mathematics that Halmos was instrumental in creating). Also, the author has attempted to conform with Halmos' tenet that symbols should, whenever possible, be substituted by words, in order to assist the reader's assimilation of the material. (An attempt to write a mathematical article subject to this constraint will convince the reader of the effort Halmos put into his writing!)

Although Halmos has done some work in each of probability theory (his Ph.D. thesis was written under the guidance of the celebrated probabilist J.L. Doob), statistics (along with L.J. Savage, he proved an important result on sufficient statistics), ergodic theory and algebraic logic, his preferred area of research (where he eventually 'settled down') was undoubtedly operator theory (more specifically, the study of bounded operators on Hilbert space). Most of his research work revolved around the so-called *invariant subspace problem*, which asks: does every bounded (=continuous) linear operator on a Hilbert space admit a non-trivial invariant subspace, meaning: is there a closed subspace, other than the zero subspace and the whole space (the two extreme trivial ones) which is mapped into itself by the operator? (Recall that a Hilbert space means a vector space over the field of complex numbers which is equipped with an inner product and is complete with respect to the norm arising from the inner product.) The answer is negative over the field of real numbers (any rotation in the plane yielding a counterexample), and is positive in the finite-dimensional complex case (thanks to complex matrices having complex eigenvalues).

The first progress towards the solution of this problem came when von Neumann showed that if an operator is compact (i.e., if it maps the unit ball into a compact set, or equivalently, if it is uniformly approximable on the unit ball by operators with finite-dimensional range), then it does indeed have a non-trivial invariant subspace. This was later shown, by Aronszajn and Smith, to continue to be true for compact operators over more general Banach, rather than just Hilbert, spaces.

Then Smith asked, and Halmos publicised, the question of whether an operator whose square is compact had invariant subspaces. It was subsequently shown by

Bernstein and Robinson, using methods of 'non-standard analysis', that if some non-zero polynomial in an operator is compact, then it has invariant subspaces. Very shortly later, Halmos came up with an alternative proof of this result, using standard methods of operator theory.

Atempting to isolate the key idea in the proof of the Aronszajn-Smith theorem, Halmos identified the notion of *quasitriangular* operators. 'Triangular' operators - those which possess an upper triangular matrix with respect to some orthonormal basis - may also be described (since finite-dimensional operators are triangular) as those which admit an increasing sequence of finite-dimensional invariant subspaces whose union is dense in the Hilbert space. Halmos' definition of quasitriangularity amounts to weakening 'invariant' to 'asymptotically invariant' in the previous sentence. An entirely equivalent requirement, as it turns out, is that the operator is of the form 'triangular + compact'; and the search was on for invariant subspaces of quasitriangular operators. This was until a beautiful 'index-theoretic' characterisation of quasitriangularity was obtained by Apostol, Foias and Voiculescu, which had the unexpected consequence that if an operator or its adjoint is not quasitriangular, then it has a non-trivial invariant subspace.

There is a parallel story involving *quasidiagonality*, also starting with a definition of Halmos and ending with a spectacular theorem of Voiculescu. Recall that in finite-dimensional Hilbert spaces, according to the spectral theorem, self-adjoint operators have diagonal matrices with respect to some orthonormal basis, and two self-adjoint operators are unitarily equivalent precisely when they have the same eigenvalues (i.e., diagonal entries in a diagonal form) which occur with the same multiplicity. Thus the spectrum of an operator ($sp\ (T) = \{\lambda \in \mathbb{C} : (T-\lambda)$ is not invertible$\}$) and the associated spectral multiplicity (the multiplicity of λ is the dimension of the null space of $T-\lambda$) form a complete set of invariants for unitary equivalence in the class of self-adjoint operators.

In the infinite-dimensional case, Hermann Weyl proved that the so-called 'essential spectrum' of a self-adjoint operator is left unchanged when it is perturbed by a compact operator; (here, the 'essential spectrum' of a self-adjoint operator is the complement, in the spectrum, of 'isolated eigenvalues of finite multiplicity';) while von Neumann showed that the essential spectrum is a complete invariant for 'unitary equivalence modulo compact perturbation' in the class of self-adjoint operators. Thus, if one allows compact perturbations, spectral multiplicity is no longer relevant. It follows that self-adjoint operators are expressible in the form 'diagonal + compact'; von Neumann even proved the strengthening with 'compact' replaced by 'Hilbert-Schmidt'. (Recall that an operator T is said to be a Hilbert-Schmidt operator if $\sum \|Te_n\|^2 < \infty$ for some (equivalently every) orthonormal basis $\{e_n\}$ of the Hilbert space.)

Halmos asked if both statements had valid conterparts for normal operators; specifically, does every normal operator admit a decomposition of the form (a) diagonal + Hilbert-Schmidt, and less stringently (b) diagonal + compact. Both questions were shown to have positive answers as a consequence of the brilliant 'non-commutative Weyl von Neumann theorem' due to Voiculescu (about representations of C^*-algebras, which specialises in the case of commutative C^*-algebras to the desired statements about normal operators); however (a) had also been independently settled by I.D. Berg.

There were two other major contributions to operator theory by Halmos: subnormal operators and unitary dilations. Both were born of his unwavering belief that the secret about general operators lay in their relationship to normal operators. He defined a *subnormal* operator to be the restriction of a normal operator to an invariant subspace; the most striking example is the unilateral shift. (Recall that the *bilateral shift* is the clearly unitary, hence normal, operator on the bilateral sequence space $\ell^2(\mathbb{Z}) = \{f : \mathbb{Z} \to \mathbb{C} : \sum_{n\in\mathbb{Z}} |f(n)|^2 < \infty\}$ defined by the equation $(Wf)(n) = f(n-1)$. In the previous sentence, if we replace $\mathbb{Z}$ by $\mathbb{Z}_+$, the analogous equation defines the *unilateral shift* U on the one-sided sequence space $\ell^2(\mathbb{Z}_+)$, which is the prototypical isometric operator which is not unitary. It should be clear that $\ell^2(\mathbb{Z}_+)$ may be naturally identified with a subspace of $\ell^2(\mathbb{Z})$ which is invariant under W, and that the restriction of W to that subspace may be identified with U.) Halmos proved that a general subnormal operator exhibits many properties enjoyed by this first example. For instance, he showed that the normal extension of a subnormal operator is unique under a mild (and natural) minimality condition. (The minimal normal extension of the unilateral shift is the bilateral shift.) Halmos also established that the spectrum of a subnormal operator is obtained by 'filling in some holes' in the spectrum of its minimal normal extension. Many years later, Scott Brown fulfilled Halmos' hope by establishing the existence of non-trivial invariant subspaces of a subnormal operator.

More generally than extensions, Halmos also initiated the study of *dilations*. It is best to first digress briefly into operator matrices. The point is that if T is an operator on $\mathcal{H}$, then any direct sum decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ leads to an identification

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

where $T_{ij} : \mathcal{H}_j \to \mathcal{H}_i,\ 1 \leq i, j \leq 2$ are operators which are uniquely determined by the requirement that if the canonical decomosition $\mathcal{H} \ni x = x_1 + x_2,\ x_i \in \mathcal{H}_i$ is written as $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, then

$$T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} T_{11}x_1 + T_{12}x_2 \\ T_{21}x_1 + T_{22}x_2 \end{bmatrix}$$

Thus, for instance, the orthogonal projection P_1 of $\mathcal{H}$ onto $\mathcal{H}_1$ is given by

$$P_1 = \begin{bmatrix} id_{\mathcal{H}_1} & 0 \\ 0 & 0 \end{bmatrix}$$

and $T_{11} = P_1 T|_{\mathcal{H}_1}$. It is customary to call T_{11} the *compression* of T to $\mathcal{H}_1$ and to call T a *dilation* of T_{11}. (Note that if (and only if) $T_{21} = 0$, then 'compression' and 'dilation' are nothing but 'restriction' and 'extension'.) Halmos wondered, but not for long, if every operator had a normal dilation; he proved that an operator has a unitary dilation if (and only if) it is a contraction (i.e., maps the unit ball of the Hilbert space into itself).

Subsequently, Sz.-Nagy showed that every contraction in fact has a 'power dilation': i.e., if T is a contraction, then there is an operator U on some Hilbert space such that, simultaneously, U^n is a dilation of T^n for every $n \geq 0$. Halmos noticed that this established the equivalence of the following conditions:

(1) T is a contraction
(2) T^n is a contraction, for each n
(3) $\|p(T)\| \leq \sup\{|f(z)| : z \in \mathbb{D}(i.e., |z| < 1)\}$

and asked if the following conditions were equivalent:

(1) T is similar to a contraction - i,e,. there exists an invertible operator S such that $S^{-1}TS$ is a contraction
(2) $\sup_n \|T^n\| \leq K$
(3) $\|p(T)\| \leq K \sup\{|f(z)| : z \in \mathbb{D}\}$

This question, as well as generalisations with $\mathbb{D}$ replaced by more general domains in $\mathbb{C}^n$, had to wait a few decades before they were solved by Gilles Pisier using 'completely bounded maps' and 'operator spaces' which did not even exist in Halmos' time!

His influence on *operator theory* may be gauged by the activity in this area during the period between his two expository papers (see [Sel1] for these papers)

Ten problems in Hilbert space, Bull of AMS, **76**, 887-933, (1970)

and

Ten years in Hilbert space, Integral Eqs. and Operator theory, , 529-564, (1979)

He listed 10 open problems in the area in the first paper, and reviewed the progress made in the second paper. While concluding the latter paper, he writes:

I hope that despite its sins of omission, this survey conveyed the flavor and the extent of progress in the subject during the last decade.

Likewise, I hope I have been able to convey something of the brilliance of the expositor in Halmos and the excitement and direction he brought to operator theory in the latter half of the last century.

Although the above account mainly discusses Halmos' contributions to operator theory, undoubtedly due to limitations of the author's familiarity or otherwise with the areas in which Halmos worked, it would be remiss on the author's part to not make at least passing mention of his contributions to Ergodic theory.

He wrote the first English book on ergodic theory (the first book on the subject being Hopf's (in German)). He made his influence felt in the field through the problems he popularised and the investigations he undertook. For instance, he gave a lot of publicity, through his book, to the question of whether a 'non-singular' transformation of a measure space - i.e., one which preserved the class of sets of measure zero - admitted an equivalent σ-finite measure which it preserved. This led to the negative answer by Ornstein, subsequent results in the area by Ito, Arnold, etc., and culminated in the very satisfying results by Krieger on *orbit equivalence*. Other contributions of his include the consideration of topologies on the set of measure-preserving transformations of a measure space (influenced no doubt by the 'category-theoretic' results obtained by Oxtoby and Ulam about the ampleness of ergodic homeomorphisms among all homoemorphisms of a cube in n dimensions) and initiating the search for square roots (and cube roots, etc.) of an ergodic transformation.

A brief biography:

A brief non-mathematical account of his life follows. (For a more complete and eminently readable write-up which serves the same purpose (and much more attractively, with numerous quotes of Halmos which serve to almost bring him to life), the reader is advised to look at the web-site:

http://scidiv.bcc.ctc.edu/Math/Halmos.html

Halmos' life is far from 'routine' - starting in Hungary and quickly moving to America. The following paragraph from his autobiographical book *I want to be a mathematician - an automathography* contains a very pithy summing up of his pre-America life:

My father, a widower, emigrated to America when I, his youngest son, was 8 years old. When he got established, he remarried, presented us with two step-sisters, and began to import us: first my two brothers, and later, almost immediately after he became a naturalized citizen, myself. In view of my father's citizenship I became an instant American the moment I arrived, at the age of 13.

The automathography referred to above contains other vignettes where we can see the problems/difficulties the young Halmos faced in coping with an alien langage and culture and a periodically unfriendly 'goddam foreigner' attitude.

After a not particularly spectacular period of undergraduate study, he began by studying philosophy and mathematics, hoping to major in the former. Fortunately for thousands of people who learnt linear algebra, measure theory and Hilbert space theory through his incomparable books, he fared poorly in the oral comprehensive exam for the masters' degree, and switched to mathematics as a major. It was only later, when he interacted with J.L. Doob that he seems to have become aware of the excitement and attraction of mathematics; and wrote a thesis on *Invariants of Certain Stochastic Transformations: The Mathematical Theory of Gambling Systems.*

After he finished his Ph.D. in 1938, he "typed 120 letters of application, and got two answers: both NO." *"The U of I took pity on me and kept me on as an instructor."* In the middle of that year a fellow graduate student and friend (Warren Ambrose) of Halmos received a fellowship at the Institute for Advanced Study. *"That made me mad. I wanted to go, too! So I resigned my teaching job, borrowed \$1000 from my father, [wrangled] an unsupported membership (= a seat in the library) at the Institute, and moved to Princeton.*

There, he attended courses, including the one by John von Neumann (*Everybody called him Johnny*) on 'Rings of Operators'. von Neumann's official assistant who was more interested in Topology, showed von Neumann the notes Halmos was taking of the course, and Halmos became the official note-taker for the course and subsequently became von Neumann's official assistant. The next year, "*with no official pre-arrangement, I simply tacked up a card on the bulletin board in Fine Hall saying that I would offer a course called "Elementary theory of matrices", and I proceeded to offer it.*" About a dozen students attended the course, some took notes and these notes were subsequently pruned into what became *Finite-dimensional vector spaces*; and Halmos' career and book-writing skills were off and running.

As a personal aside, this book was this author's first introduction to the charm of abstract mathematics, and which prompted him to go to graduate school at Indiana University to become Halmos' Ph.D. student - his last one as it turned out.

This author cannot begin to enumerate all the things he learnt from this supreme teacher, and will forever be in his debt.

As a final personal note, I should mention Virginia, his warm and hospitable wife since 1945. I remember going to their house for lunch once and finding Paul all alone at home; his grumbled explanation: “Ginger has gone cycling to the old folks home, to read to some people there, who are about 5 years younger than her!”. (She was past 70 then.) She still lives at Los Gatos, California. They never had children, but there were always a couple of cats in their house.

It seems appropriate to end with this quote from the man himself:

"I'm not a religious man, but it's almost like being in touch with God when you're thinking about mathematics."

References:

[Sel1] *P.R. Halmos: Selecta - Research Mathematics*, Springer-Verlag, New York (1983).

[Sel2] *P.R. Halmos: Selecta - Expository Writing*, Springer-Verlag, New York (1983).

Suggested reading:

Technical papers by Halmos:

1. *Ten Problems in Hilbert space*, Bulletin of the AMS, **76**, no. 5, 887-993, 1970.
2. *Ten Years in Hilbert space*, Integral Eqs. and Operator Theory, **2/4**, 393-428, 1979.

Expository articles by Halmos:

1. *How to write mathematics*, l'Enseignement matheématique, **XVI**, no. 2, 123-152, 1970.
2. *How to write mathematics*, Notices of the AMS, **23**, no. 4, 155-158, 1974.
3. *The Teaching of Problem Solving*, Amer. Math. Monthly, **82**, No. 5, 466-470, 1975.

Two books non-technical books by Halmos:

1. *I want to be a mathematician - an Automathography*, Springer-Verlag, New York, (1985).
2. *I have a photographic memory*, AMS, Providence, RI, (1987).

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