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ON JONES' PLANAR ALGEBRAS

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ABSTRACT

We show that a certain natural class of tangles 'generate the collection of all tangles with respect to composition'. This result is motivated by, and describes the reasoning behind, the 'uniqueness assertion' in Jones' theorem on the equivalence between extremal subfactors of finite index and what we call 'subfactor planar algebras' here. This result is also used to identify the manner in which the planar algebras corresponding to $M \subset M_1$ and $N^{op} \subset M^{op}$ are obtained from that of $N \subset M$.

Our results also show that 'duality' in the category of extremal subfactors of finite index extends naturally to the category of 'general' planar algebras (not necessarily finite-dimensional or spherical or connected or C^*, in the terminology of Jones).

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1. Introduction

The last two decades have witnessed a dramatic rise in the systematic use of topological and pictorial methods in tackling algebraic problems - see, for example, [K1], [K2], [Li1], [Li2]. This paper is primarily an attempt to give an exposition of some of the notions that go into the structures introduced by Vaughan Jones (in [J1]), which he called 'general planar algebras'. He showed, using earlier work of Popa ([Po2]), that 'good planar algebas' (which he calls C^*-planar algebras) correspond to 'good subfactors' (the so-called extremal ones of finite index).

Jones also showed that the planar algebra corresponding to a subfactor was uniquely determined by certain requirements. We wanted to understand this 'uniqueness assertion' as well as the passage from the planar algebra (say P) corresponding to a subfactor $N \subset M$, to the planar algebra (which we denote by ${}^-P$ in this paper) corresponding to the 'dual subfactor' $M \subset M_1$; and this paper is the result of that endeavour.

To start with, we adopt the 'operadic' approach, as against the equivalent 'generators-and-relations' approach to planar algebras that Jones first espoused in [J1]. We start with a somewhat lengthy section (§2) devoted to establishing the terminology and notation that we follow later, and a description of Jones' result alluded to earlier. This section is almost exactly the same as a similar preliminary section in [KLS]. One reason for this repetition is that we need that notation to formulate much of the later sections; another reason is that our notation and terminology are slightly different from Jones. (Thus, we simply refer to his 'general planar algebras' as 'planar algebras'; we use the expression 'subfactor planar algebra' to describe the 'good planar algebras' from the point of view of Jones' theorem; we multiply from top to bottom as with multiplication of braids, rather than from bottom to top as in [J1] and [BJ]; etc.)

The next section (§3) is devoted to the result, described in the opening sentence of the abstract, concerning 'generating sets of tangles'.

In §4, we discuss a fairly simple-minded approach to constructing 'new planar algebras from old' via what we call 'operations on tangles'. (This may be viewed as formalising some constructions described briefly in [J1].)

2. Planar algebras

This section is devoted to a survey of such facts about planar algebras as we will require. (See [J1] - where these objects first appear - and [J2] - where the operadic approach is discussed - and also [La] for a 'crash-course'.)

We define a set *Col*, whose members we shall loosely call 'colours', by

$$Col \;=\; \{0_+, 0_-, 1, 2, 3, \cdots\} \tag{2.1}$$

Some of the basic objects here are the so-called k-tangles (where $k \in Col$), towards whose definition we now head.

Consider a copy D_0 of the closed unit disc $D = \{z \in \mathbb{C} : |z| \leq 1\}$, together with a collection $\{D_i : 1 \leq i \leq b\}$ of some number b (which may be zero) of pairwise disjoint (adequately compressed) copies of D in the interior of D_0. Suppose now that we have a pair (T, f), where:

(a) T is an oriented compact one-dimensional submanifold of $D_0 \backslash \cup_{i=1}^{b} Int(D_i)$ - where 'Int' denotes interior - with the following properties:

(i) $\partial(T) \subset \cup_{i=0}^{b} \partial(D_i)$ and all intersections of T with ∂D are transversal;

(ii) each connected component of the complement of T in $Int(D_0) \backslash \cup_{i=1}^{b} Int(D_i)$ comes equipped with an orientation which is consistent with the orientation of T;

(iii) $|T \cap \partial(D_i)| = 2k_i$ for some integers $k_i \geq 0$, for each $0 \leq i \leq b$; and

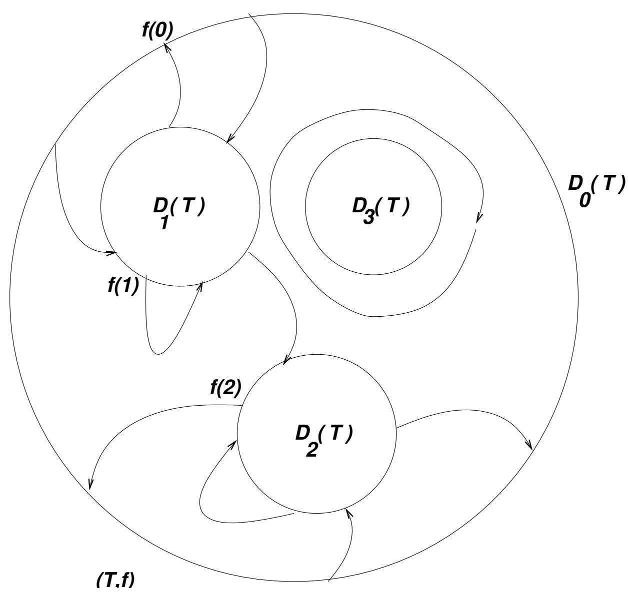
(b) f specifies some 'distinguished points' thus: whenever $0 \leq i \leq b$ is such that $k_i > 0$, we are given a 'distinguished point' $f(i) \in T \cap \partial(D_i)$; these distinguished points (which we will sometimes follow Jones and simply denote by $*$) are required to satisfy the following 'compatibility condition' with respect to the orientation of (a)(ii):

the component of T which contains $f(i)$ is required to be oriented (at $f(i)$) away from or towards ∂D_i according as $i > 0$ or $i = 0$.

The orientation requirement above ensures that there is a unique *chequerboard shading* of $Int(D_0) \setminus (\cup_{i=1}^b Int(D_i) \cup T)$ as follows: shade a component white or black according as it is equipped with the mathematically positive or negative orientation in (a)(ii) above. Thus, whenever one moves along any component of T in the direction specified by its orientation, the region immediately to one's right is shaded black. With the above notation, there are two possibilities, for each $0 \leq i \leq b$: (i) $k_i > 0$, in which case we shall say that D_i is of colour k_i; and (ii) $k_i = 0$; in this case, we shall say that D_i is of colour $0_\pm$ according as the region immediately adjacent to ∂D_i is shaded white or black in the 'chequerboard shading'.

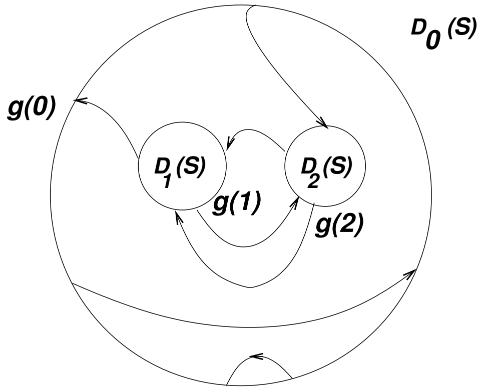
We shall consider two such pairs (T_i, f_i) to be *equivalent* if the T_i are isotopic via an isotopy which preserves the orientation and the 'distinguished points'. Finally, an equivalence class as above is called a **k-tangle** where k is the colour of the external disc D_0.

An example of a 3-tangle with 3 internal discs is illustrated here - in which $b = 3$ and the internal discs D_1, D_2 and D_3 have colours $3, 3$ and 0_- respectively:

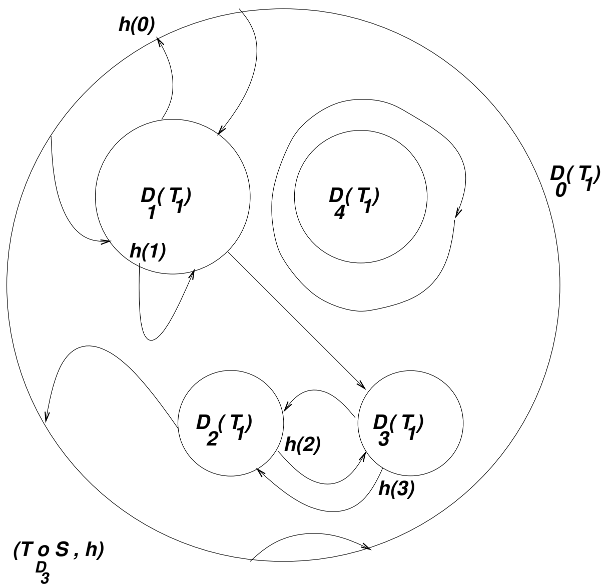


There is a natural way to 'compose' two tangles. For instance, suppose (T, f) is a k-tangle, with $b \geq 1$ internal discs; if one of these internal discs D_i has colour k_i, and if (S, g) is a k_i-tangle, then $T \circ_{D_i} S$ is the k-tangle obtained by 'glueing S into D_i' (taking care to attach $g(0)$ to $f(i)$ in case $k_i > 0$).

For example, if (T, f) is as above and if (S, g) is the 3-tangle given by



then a possible 'composite tangle' is the 3-tangle $(T_1 = T \circ_{D_2} S, h)$ given thus:



The collection of 'coloured tangles' with the 'composition' defined above is referred to as the *(coloured) planar operad*; and by a **planar algebra** is meant an 'algebra over this operad': in other words, a planar algebra P is a family $P = \{P_k : k \in Col\}$ of vector spaces with the following property: for every $k_0 = k_0(T)$-tangle (T, f) with $b = b(T)$ internal discs $D_1(T), \cdots, D_{b(T)}(T)$ of colours $k_1(T), \cdots, k_{b(T)}(T)$, there is associated a linear map

$$Z_T : \otimes_{i=1}^{b} P_{k_i(T)} \to P_{k_0(T)}$$

which is 'compatible with composition of tangles' in the following obvious manner.

If $1 \leq i \leq b$ is fixed, and if (S, g) is a $k_i(T)$-tangle with $b(S)$ internal discs - call them $D_1(S), \cdots, D_{b(S)}(S)$ - with colours $k_1(S), \cdots, k_{b(S)}(S)$ (say) - then we know that the composite tangle $T_1 = T \circ_{D_i(T)} S$ is a $k_0(T)$-tangle with the $(b(T) + b(S) - 1)$ internal discs given by

$$D_j(T_1) = \begin{cases} D_j(T) & \text{if } 1 \leq j < i \\ D_{j-i+1}(S) & \text{if } i \leq j \leq i + b(S) - 1 \\ D_{j-b(S)+1}(T) & \text{if } i + b(S) \leq j \leq b(T) + b(S) - 1 \end{cases} \quad ;$$

it is required that the following diagram commutes:

$$\begin{tikzcd}
(\otimes_{j=1}^{i-1} P_{k_j(T)}) \otimes (\otimes_{j=1}^{b(S)} P_{k_j(S)}) \otimes (\otimes_{j=i+1}^{b(T)} P_{k_j(T)}) \arrow[dd, "id \otimes Z_S \otimes id"'] \arrow[dr, "Z_{T \circ_{D_i(T)} S}"] & \\
& P_{k_0(T)} \\
\otimes_{j=1}^{b(T)} P_{k_j(T)} \arrow[ur, "Z_T"'] &
\end{tikzcd} \tag{2.2}$$

We shall also demand of our planar algebras that the assignment $T \mapsto Z_T$ is 'independent of the ordering of the internal discs of T' in the following sense: if we think of Z_T as returning an output (in $P_{k_0(T)}$) for every choice of an input x_i (from $P_{k_i(T)}$) for each internal disc D_i, then the output should be independent of 'the ordering of the internal discs'. Explicitly, this may be stated thus: for a tangle T with b internal discs, and for a permutation $\sigma \in S_b$, let us write U_σ for the map from any tensor-product $V_1 \otimes \cdots \otimes V_b$ of b vector spaces to the tensor product $V_{\sigma^{-1}(1)} \otimes \cdots \otimes V_{\sigma^{-1}(b)}$ defined by

$$U_\sigma(\otimes_{i=1}^b v_i) = \otimes_{i=1}^b v_{\sigma^{-1}(i)} \ .$$

Let us further define $\sigma(T)$ to be the tangle which differs from T only in the numbering of its internal discs, this numbering being given by $D_i(\sigma(T)) = D_{\sigma^{-1}(i)}(T)$, $1 \leq i \leq b(T)$. Our requirement of 'independence of the ordering of the internal discs' is that

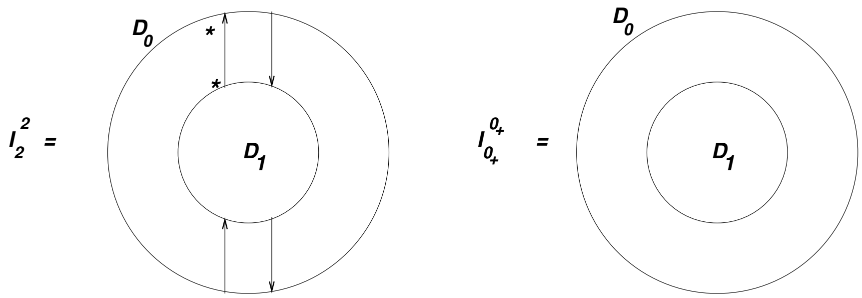
$$Z_{\sigma^{-1}(T)} = Z_T \circ U_\sigma \ . \tag{2.3}$$

Strictly speaking, we need to exercise a little caution when 0 is involved. For instance, in order to make sense of the domain of Z_T, when the tangle T has no internal discs (i.e., $b(T) = 0$), we need to adopt the convention that the empty tensor product is the underlying field, which we shall always assume is $\mathbb{C}$ (but which is not essential for the non- C^* case). So each P_k has a distinguished subset, *viz.,* $\{Z_T(1) : T$ a k-tangle without internal discs $\}$.

Next, our statement of the 'compatibility requirement (2.2)' needs to be slightly modified if the tangle S has no internal discs; thus, if $b(S) = 0$, the requirement (2.2) needs to be modified thus:

$$\begin{array}{ccc} \otimes_{j\neq i} P_{k_j(T)} & & \\ \cong \ \downarrow & Z_{T\circ_{D_i(T)}S} \ \searrow & \\ (\otimes_{j<i}P_{k_j(T)})\otimes\mathbb{C}\otimes(\otimes_{j<i}P_{k_j(T)}) & & P_{k_0(T)} \\ id\otimes Z_S\otimes id \ \downarrow & Z_T \ \nearrow & \\ \otimes_{j=1}^{b} P_{k_j(T)} & & \end{array} \tag{2.4}$$

Further, we need to make an additional assumption in order to 'rule out some degeneracies'. To see this, consider the k-tangles I_k^k[1], $k \in Col$, with one internal disc also of colour k - thus, in our notation, $b(I_k^k) = 1, k_0(I_k^k) = k_1(I_k^k) = k$ - defined as in the figure below:



(The understanding is that $I_{0_+}^{0_+}$ consists of the 'empty submanifold of $D_0 \setminus D_1$' and that the annular region $D_0 \setminus D_1$ is equipped with the 'mathematically positive orientation' and hence shaded white in the chequerboard shading. In the case of $I_{0_+}^{0_+}$, the only difference is that the annular region is shaded black.)

It is easily seen that, for every k, and for every k-tangle T, we have $I_k^k \circ_{D_1} T = T$, and hence

$$Z_{I_k^k} \circ Z_T = Z_T \ . \tag{2.5}$$

It follows that $Z_{I_k^k}$ is an idempotent endomorphism of P_k whose range contains the range of Z_T for every k-tangle T.

The *non-degeneracy condition* we wish to impose is that P_k is spanned by the ranges of the Z_T's, as T ranges over all k-tangles; in view of the above comments, this is equivalent to the following condition, which we shall henceforth assume is satisfied by all our planar algebras:

$$Z_{I_k^k} = id_{P_k} \ \forall \ k \in \ Col. \tag{2.6}$$

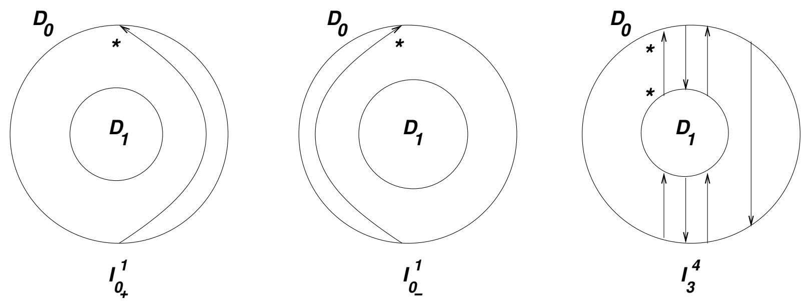
Thus, what we shall mean by a planar algebra is a collection $P = \{P_k : k \in \ Col\}$ of vector spaces, equipped with an assignment $T \mapsto Z_T$ of multilinear

[1] In the sequel we shall consistently use the convention of writing T_n^m to denote a tangle T with $b(T) = 1, k_0(T) = m, k_1(T) = n$; so for instance we might have $(T_1)_n^m = T_k^m \circ S_n^k$ as with matrix multiplication.

maps to coloured tangles, in such a manner that equations (2.3), (2.2), (2.4) and (2.6) are satisfied.

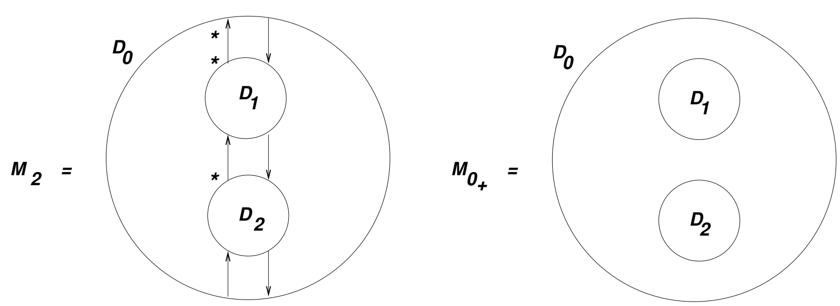
We shall need the following tangles:

The inclusion tangles: For every $k \in Col$, there is an associated $(k+1)$-tangle I_k^{k+1} with one internal disc of colour k - where of course $0_\pm + 1 = 1$; rather than giving the formal definition, we just illustrate $I_{0_+}^1, I_{0_-}^1$ and I_3^4 below - the idea being that an 'extra vertical line is stuck on to the far right (in all but one exceptional case)'.



It should be clear that $Z_{I_k^{k+1}} : P_k \to P_{k+1}$. It will turn out that these 'inclusion' tangles indeed induce injective maps in the case of 'good' planar algebras (the ones with a 'non-zero modulus').

The multiplication tangles: For each $k \in Col$, these are k-tangles M_k with two internal discs, both of colour k, which equip P_k with a multiplication. We illustrate the cases $k = 2$ and $k = 0_+$ below:



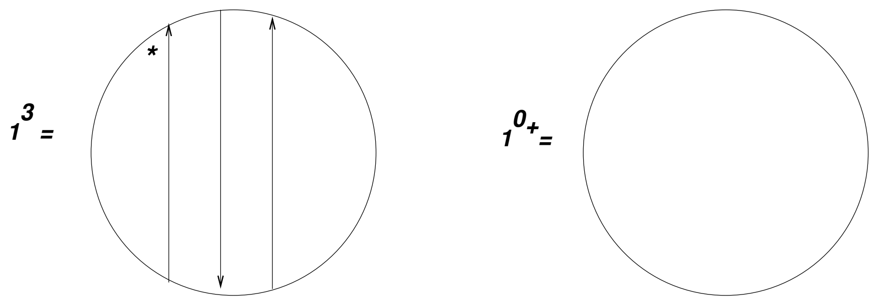
(As in the case of the 'identity annular tangles $I_{0_\pm}^{0_\pm}$', the tangles $M_{0_\pm}$ consist only of the empty submanifold (of $D_0 \setminus \cup_{i=1}^2 Int(D_i)$), the only distinction between $M_{0_\pm}$ being that the region $D_0 \setminus \cup_{i=1}^2 Int(D_i)$ is shaded white and black in M_{0_+} and M_{0_-}, respectively.)

It is easy to see that each P_k is an associative algebra, with respect to multiplication being defined by

$$x_1 x_2 = Z_{M_k}(x_1 \otimes x_2) \ .$$

It must be noted that this convention - of putting the first factor in the disc on top - is opposite to the one adopted in [BJ], for instance; and also that $P_{0_\pm}$ are even commutative.

We also wish to point out that the fact that the P_k's are unital algebras is a consequence of our 'non-degeneracy condition' and of the compatibility condition (2.4): in fact, consider the k-tangle 1^k, which has no internal discs, defined analogous to the case 1^3 illustrated below: (The tangle 1^{0+} is again the empty submanifold of D_0, with the interior of D_0 shaded white; and 1^{0-} is defined analogously except that 'white' is replaced by 'black'.)

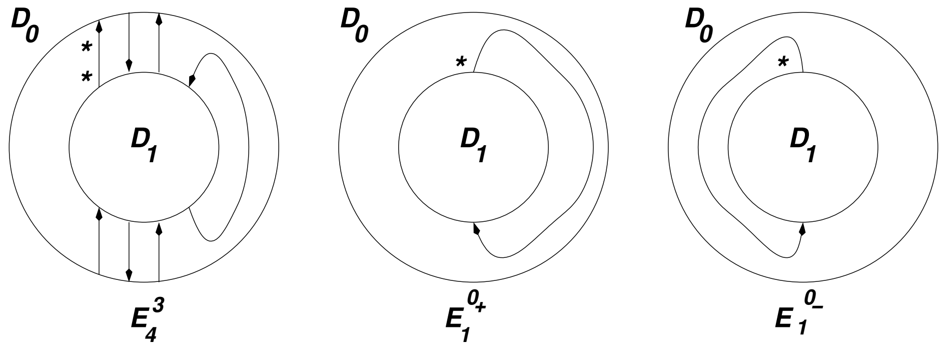


Notice that $M_k \circ_{D_2} 1^k = I_k^k$, and if we write $1_k = Z_{1^k}(1)$ (where the 1 on the right is the 1 in $\mathbb{C}$), then we may deduce from (2.4) that for arbitrary $x \in P_k$:

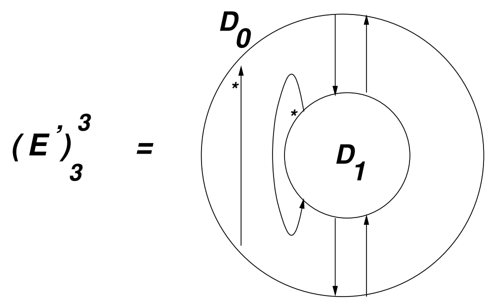
$$\begin{aligned} x \cdot 1_k &= Z_{M_k}(x \otimes 1_k) \\ &= Z_{M_k}(x \otimes Z_{1^k}(1)) \\ &= Z_{M_k \circ_{D_2} 1^k}(x) \\ &= Z_{I_k^k}(x) \\ &= x \ . \end{aligned}$$

A similar argument - with D_2 replaced by D_1 shows that 1_k is also a 'left-identity'. Hence P_k is a unital associative algebra with 1_k as the multiplicative identity. A similar argument also shows that the 'inclusion tangles' in fact induce homomorphisms of unital algebras - so that, in 'good cases', any planar algebra admits the structure of an associative unital algebra which is expressed as an increasing union of subalgebras.

The conditional expectation tangles: These are two families of tangles $\{E_{k+1}^k : k \in Col\}$, and $\{(E')_k^k : k \geq 1\}$, where (by our notational convention for 'annular tangles') (i) E_{k+1}^k is a k-tangle with one internal disc of colour $k+1$, which is defined by 'capping off the last strand'; again, rather than giving a formal definition, we illustrate E_4^3, E_1^{0+} and E_1^{0-} below:



and (ii) $(E')^k_k$ is a k-tangle with one internal disc of colour k, which is defined by 'capping to the left'; again, rather than giving a formal definition, we illustrate $(E')^3_3$ below:



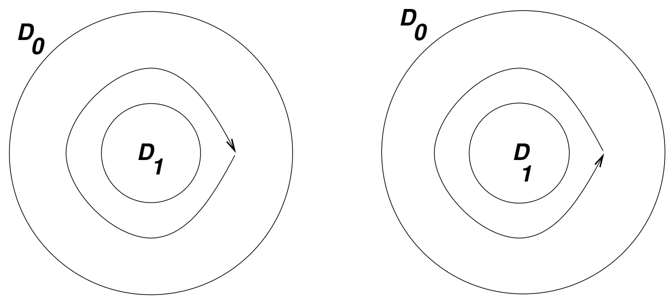
Clearly $Z_{E^k_{k+1}} : P_{k+1} \to P_k$ while $Z_{(E')^k_k} : P_k \to P_k$. (In fact, the range of $Z_{(E')^k_k}$ is contained in $Z_{I^k_{k-1}} \circ \cdots \circ Z_{I^2_1}(P_1)' \cap P_k$.)

Some planar algebras may have various additional good features, which we now list, so as to be able to state Jones' result on planar algebras and subfactors.

Connectedness: A planar algebra P is said to be *connected* if *dim* $P_{0_\pm} = 1$.

Since $P_{0_\pm}$ are unital $\mathbb{C}$-algebras, it follows that if P is connected, then there exist unique algebra isomorphisms $P_{0_\pm} \cong \mathbb{C}$; they will necessarily identify what we called $1_{0_\pm}$ (recall the 1_k above) with $1 \in \mathbb{C}$.

For the next definition, we need to introduce two more tangles. Consider the annular tangles $T^\pm_\mp$ of colours $0_\pm$, with an internal disc of colour $0_\mp$, given as follows:



Modulus: A connected planar algebra P is said to have *modulus* δ if there exists a scalar δ such that $Z_{T^{\pm}_{\mp}}(1) = \delta \; 1_{\mathbb{C}}$. We will primarily be interested in the case when the modulus is positive.

It must be noted that if P has modulus δ, then

$$Z_{E^k_{k+1}} \circ Z_{I_k^{k+1}} = \delta \; id_{P_k} \; \forall k \in \; Col \; ;$$

and in particular, if $\delta \neq 0$, then the 'inclusion tangles' do induce injective maps.

Finite-dimensionality: A planar algebra P is said to be finite-dimensional if $dim \; P_k \; < \infty \; \forall k \in \; Col$.

Suppose P is a connected planar algebra and T is a 0-tangle (by which we shall mean a 0_+- or a 0_--tangle). If T has internal discs D_i of colour k_i, and if $x_i \in P_{k_i}$ for $1 \leq i \leq b$, then $Z_T(\otimes_{i=1}^b x_i) \; \in \mathbb{C}$, where we have made the canonical identifications $P_{0_\pm} = \mathbb{C}$. This assignment of scalars to 'labelled 0-tangles' is also referred to as the *partition function associated to the planar algebra.*

Sphericality: A planar algebra is said to be *spherical* if its partition function assigns the same value to any two 0-tangles which are isotopic as tangles on the 2-sphere (and not just the plane).

The last bit of terminology we will need is that of the 'adjoint of a tangle'. Suppose (T, f) is a k_0-tangle as defined earlier, with external disc D_0 and b internal discs D_i of colours k_i. We then define its adjoint to be the k_0-tangle (T^*, f^*) given thus:

(a) Let ϕ be any orientation reversing smooth map of $D_0(T)$ onto a disc D_0^* and let T^* be defined by requiring that $D_i(T^*) = \phi^*(D_i(T)), \; 0 \leq i \leq b(T)$, and its underlying one-submanifold of D_0^* is $\phi(T)$, with the orientation - of T^* as well as the components of its complement in $(D_0(T^*) \setminus \cup_{i=1}^b D_i(T^*))$ - being opposite to the one inherited via ϕ. (In other words, a region $\phi(R)$ - in the complement of T^* in $(D_0(T^*) \setminus \cup_{i=1}^b D_i(T^*))$ - has the same colour as R in the chequerboard shading.)

(b) If $k_i > 0$, define $\widetilde{f(i)}$ to be the 'first point' of $T \cap \partial(D_i(T))$ that is encountered as one proceeds anti-clockwise along $\partial(D_i(T))$ from $f(i)$; and define $f^*(i) = \phi(\widetilde{f(i)})$.

Finally, we shall say that P is a *subfactor planar algebra* if:

(i) P is connected, finite-dimensional, spherical, and has positive modulus;

(ii) each P_k is a C^*-algebra in such a way that, if (T, f) is a k_0-tangle as above, with external disc D_0 and b internal discs D_i of colours k_i, and if $x_i \in P_{k_i},\ 1 \leq i \leq b$, then

$$Z_T(x_1 \otimes \cdots \otimes x_b)^* \ = \ Z_{T^*}(x_1^* \otimes \cdots \otimes x_b^*) \ ;$$

and

(iii) if we define the 'pictorial trace' on P by

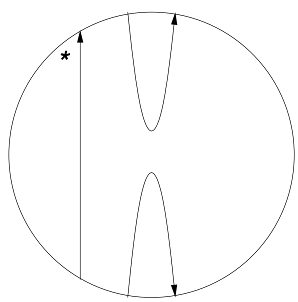
$$tr_{k+1}(x) \ 1_+ \ = \ \delta^{-k-1} Z_{E_1^{0+}} Z_{E_2^1} \cdots Z_{E_{k+1}^k}(x) \qquad (2.7)$$

for $x \in P_{k+1}$, then tr_m is a faithful positive trace on P_m for all $m \geq 1$.

It should be obvious that if P is a subfactor planar algebra, the 'tr_m's are consistent and yield a 'global trace tr on P'.

Our primary interest in planar algebras stems from a beautiful result - Theorem 2.1 below - of Jones' (see [J1]). Before stating it, it will be convenient for us to introduce another family $\{\mathcal{E}^k : k \geq 2\}$, of tangles, where $\mathcal{E}^k$ is a k-tangle with no internal discs; we illustrate the case $k = 3$ below:

k = 3



THEOREM 2.1 *Let*

$$N \subset M (= M_0) \subset^{e_1} M_1 \subset \cdots \subset^{e_k} M_k \subset^{e_{k+1}} \cdots$$

be the tower of the basic construction associated to an extremal subfactor with $[M : N] = \delta^2 < \infty$. Then there exists a unique subfactor planar algebra $P = P^{N \subset M}$ of modulus δ satisfying the following conditions:

*(0) $P_k^{N \subset M} = N' \cap M_{k-1} \ \forall k \geq 1$ - where this is regarded as an equality of *-algebras which is consistent with the inclusions on the two sides;*

(1) $Z_{\mathcal{E}^{k+1}}(1) = \delta\ e_k\ \forall\ k \geq 1$*;*

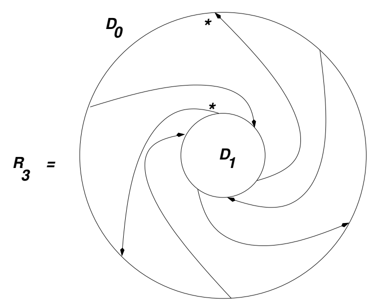
(2) $Z_{(E')^k_k}(x) = \delta\ E_{M' \cap M_{k-1}}(x)\ \forall\ x \in N' \cap M_{k-1},\ \forall k \geq 1$*;*

(3) $Z_{E^k_{k+1}}(x) = \delta\ E_{N' \cap M_{k-1}}(x)\ \forall\ x \in N' \cap M_k$*; and this is required to hold for all* k *in* Col*, where for* $k = 0_\pm$*, the equation is interpreted as*

$$Z_{E_1^{0_\pm}}(x) \ = \ \delta\ tr_M(x)\ \forall\ x \in N' \cap M\ .$$

Conversely, any subfactor planar algebra P *with modulus* δ *arises from an extremal subfactor of index* δ^2 *in this fashion.*

REMARK 2.2 We want to single out one specific class of tangles which play a very important role in the proof of the above theorems as well as in the general theory. These are a family $\{R_k : k \geq 2\}$ of tangles which will occur frequently in the sequel; so we shall, in the interest of convenience, drop our (otherwise) standing convention for annular tangles, and write R_k rather than R^k_k. Thus this *rotation tangle* R_k is the k-tangle with one internal disc of colour k, which we illustrate below for $k = 3$:



3. Generating tangles

In the interest of notational convenience, we shall, in the sequel, simply write $T \circ S$ whenever T is an *annular tangle* (i.e., $b(T) = 1$) and the composition makes sense (i.e., $k_0(S) = k_1(T)$).

This section is devoted to a proof of the following result, which is not very surprising in view of Jones' Theorem 2.1. (Of course this result yields an alternative proof of the 'uniqueness assertion' of Jones' Theorem 2.1. The 'little more' that this proof would show is that there exists a unique planar algebra - not assumed *a priori* to fulfil the further restraints imposed on a subfactor planar algebra - satisfying conditions (0)-(3) of Theorem 2.1.)

THEOREM 3.3 *Let* $\mathcal{T}$ *denote the set of all coloured tangles, and suppose* $\mathcal{T}_1$ *is a subclass of* $\mathcal{T}$ *which satisfies:*

(a) $\{1^{0_+}, 1^{0_-}\} \cup \{\mathcal{E}^k : k \geq 2\} \cup \{E^k_{k+1}, M_k, I^{k+1}_k : k \in\ Col\} \cup \{(E')^k_k : k \geq 1\} \subset \mathcal{T}_1$*;*

and

(b) $\mathcal{T}_1$ *is closed under composition, when it makes sense; i.e.,* $T, S \in \mathcal{T}_1, k_0(S) = k_i(T) \;\Rightarrow\; T \circ_{D_i(T)} S \in \mathcal{T}_1$.

Then $\mathcal{T}_1 = \mathcal{T}$.

In the sequel, we shall use the following notations:

$$\begin{aligned} I_k^l &= I_{l-1}^l \circ (\cdots (I_{k+1}^{k+2} \circ I_k^{k+1}) \cdots)) \ , \ k < l \\ E_l^k &= E_{k-1}^k \circ (\cdots (E_{l-1}^{l-2} \circ E_l^{l-1}) \cdots)) \ , \ k > l \\ R_k^{(m)} &= R_k \circ (R_k \circ \cdots (R_k \circ R_k) \cdots)) \quad (m \text{ times}) \ , \ k \geq 2, m \geq 1. \end{aligned}$$

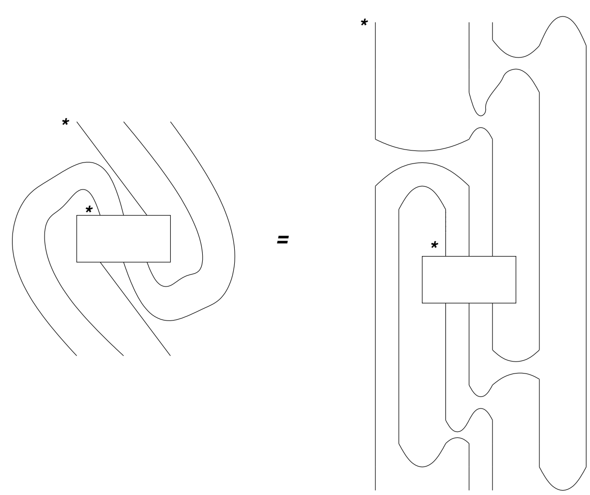
Lemma 3.4 *For any* $k \geq 2$, *we have*

$$R_k = E_{k+1}^k \circ \big\{ (M_{k+1} \circ_{D_1} T^{k+1}) \circ_{D_2} \big((E')_{k+1}^{k+1} \circ ((M_{k+1} \circ_{D_1} I_k^{k+1}) \circ_{D_2} T^{k+1}) \big) \big\} \ ,$$

where we inductively define the $T^k, k \geq 2$ *by*

$$T^k \; = \; \begin{cases} \mathcal{E}^2 & \text{if } \ k = 2 \\ (M_k \circ_{D_1} \mathcal{E}^k) \circ_{D_2} (I_{k-1}^k \circ T^{k-1}) & \text{if } \ k > 2 \end{cases}$$

Proof: The lemma follows from the definitions. We illustrate the proof with a diagram for the case $k = 3$:



□

Hence, in view of Lemma 3.4, the truth of Theorem 3.3 will follow from the following theorem, which is what we shall prove in this section.

Theorem 3.5 *Let* $\mathcal{T}$ *denote the set of all coloured tangles, and suppose* $\mathcal{T}_0$ *is a subclass of* $\mathcal{T}$ *which satisfies:*

(a) $\{1^{0_+}, 1^{0_-}\} \cup \{R_k : k \geq 2\} \cup \{E^k_{k+1}, M_k, I^{k+1}_k : k \in \ Col\} \subset \mathcal{T}_0$; *and*

(b) $\mathcal{T}_0$ *is closed under composition, when it makes sense; i.e.,* $T, S \in \mathcal{T}_0, k_0(S) = k_i(T) \ \Rightarrow \ T \circ_{D_i(T)} S \in \mathcal{T}_0$.

Then $\mathcal{T}_0 = \mathcal{T}$.

The proof of the theorem will be accomplished in a series of steps. Our first lemma will allow us in the sequel to not have to bother with tangles which 'have one or more loop in them'.

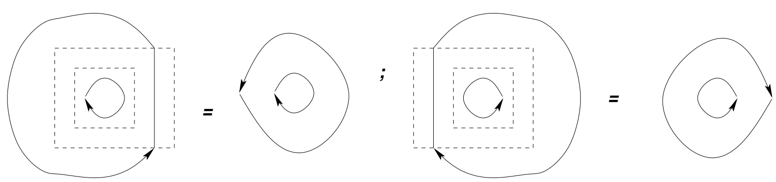
Lemma 3.6 *If* $k_0(T) = 0_\pm$ *and* $b(T) = 0$, *then* $T \in \mathcal{T}_0$; *i.e.,* $\mathcal{T}_0$ *contains all tangles of colour* $0_\pm$ *which do not have any internal discs.*

Proof: Since $1^{0_\pm}, M_{0_\pm} \in \mathcal{T}_0$, a moment's thought and a simple induction argument reveals that it suffices to prove the following assertion:

If C *is a tangle with* $k_0(C) = 0_\pm$, *and if* C_1 *is the tangle with* $k_0(C_1) = 0_\mp$ *which consists of one circle enclosing* C *in its interior, and if* $C \in \mathcal{T}_0$, *then also* $C_1 \in \mathcal{T}_0$.

The assertion above is a consequence of the relation

$$C_1 = E_1^{0_\mp} \circ (I^1_{0_\pm} \circ C) \ .$$



$\square$

Remark 3.7 *If a tangle* T *contains* l *loops, then there exists a tangle* T_1 *with an internal disc* D *of colour* $0_\pm$ *such that* T_1 *contains* $(l-1)$ *loops and* $T = T_1 \circ (T^\pm_\mp \circ 1^\mp)$ *(where* $T^\pm_\mp$ *are as in the figures just preceding the definition of 'modulus'). It follows, by induction on* l, *that if a statement is valid about all tangles in a class* $\mathcal{T}_0$ *which is closed under composition, and if* $\mathcal{T}_0$ *contains all tangles without loops, then we must have* $\mathcal{T}_0 = \mathcal{T}$. *(This is what was meant by the sentence preceding the statement of Lemma 3.6.)*

Lemma 3.8 $\mathcal{T}_0$ *contains* $\mathcal{E}^k$ *for all* $k \geq 2$.

Proof: We have

$$\mathcal{E}^k = E^k_{k+1} \circ (R^{(k)}_{k+1} \circ (I^{k+1}_{0_+} \circ 1^{0_+})) \ ,$$

for $k = 2, 3$, and

$$\begin{aligned}\mathcal{E}^{2n} &= R_{2n}^{(n)} \circ (I_2^{2n} \circ \mathcal{E}^2) \\ \mathcal{E}^{2n+1} &= E_{2n+2}^{2n+1} \circ (R_{2n+2}^{(n+1)} \circ (M_{2n+2} \circ (I_2^{2n+2} \circ \mathcal{E}^2, I_3^{2n+2} \circ \mathcal{E}^3))) \ ,\end{aligned}$$

where we write $M_k \circ (T_1, T_2)$ for $(M_k \circ_{D_1} T_1) \circ_{D_2} T_2$ for k-tangles T_1 and T_2. □

In view of Lemma 3.6, we may, and shall, assume in the rest of our proof of Theorem 3.5, that none of our tangles 'have any loops' in them. Also, in the following discussion, it will be convenient to assume that if (T, f) is a k-tangle, then the $2k_i$ marked points on $T \cap \partial D_i(T)$ are labelled $f(i) = 1, 2, \cdots, 2k_i$ 'in the clockwise order'.

LEMMA 3.9 *Suppose T is a tangle 'without any loops' - with $k_i = k_i(T)$ for $0 \leq i \leq b(T)$.*

(a) The following conditions on T are equivalent:

(i) T has a string joining the points labelled k_0 and $k_0 + 1$ on $\partial D_0(T)$;

(ii) $T = I_{k_0-1}^{k_0} \circ (E_{k_0}^{k_0-1} \circ (M_{k_0} \circ (\mathcal{E}_{k_0}, T)))$.

(b) The following conditions on T are equivalent:

(i) T has a string joining the points labelled $k_0 - 1$ and k_0 on $\partial D_0(T)$;

(ii) $T = M_{k_0} \circ (\mathcal{E}^{k_0}, I_{k_0-1}^{k_0} \circ (E_{k_0}^{k_0-1} \circ T))$.

(c) The following conditions on T are equivalent:

(i) T has a string joining the points labelled k_i and $k_i + 1$ on $\partial D_i(T)$ for some $i \geq 1$;

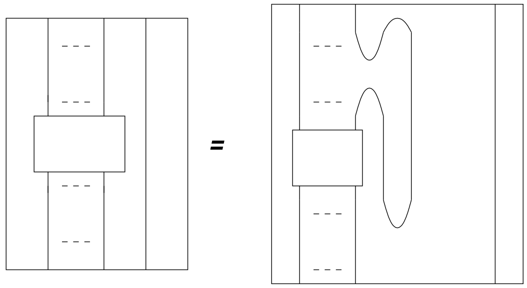
(ii) $T = (T \circ_{D_i} M_{k_i} \circ (I_{k_i-1}^{k_i}, \mathcal{E}^{k_i})) \circ_{D_i} E_{k_i}^{k_i-1}$.

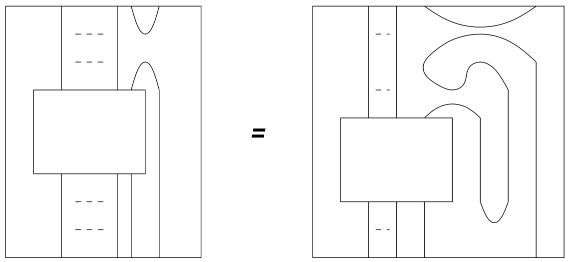
(d) The following conditions on T are equivalent:

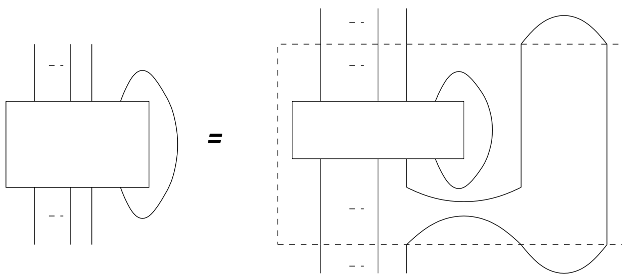
(i) T has a string joining the points labelled $k_i - 1$ and k_i on $\partial D_i(T)$ for some $i \geq 1$;

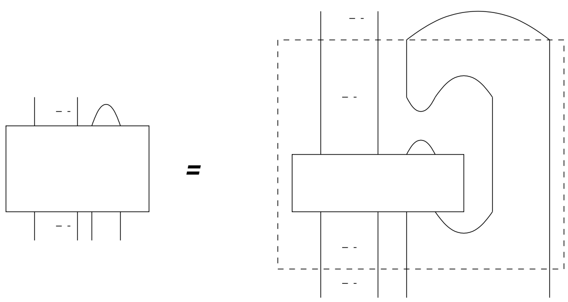
(ii) $T = (T \circ_{D_i} I_{k_i-1}^{k_i}) \circ_{D_i} (E_{k_i}^{k_i-1} \circ M_{k_i}(\mathcal{E}^{k_i}, I_{k_i}^{k_i}))$.

Proof: The proof merely involves staring at the following diagrams:

(a) 

(b) 

(c) 

(d) 

□

LEMMA 3.10 *If T is a tangle without any loops or internal discs, then $T \in \mathcal{T}_0$.*

Proof: We prove this by induction on $k_0(T)$.

We see then that, under our hypotheses,

$$T \;=\; \begin{cases} 1^{0_\pm} & \text{if} \;\; k_0(T) = 0_\pm \\ 1^1 = I^1_{0_+}(1^{0_+}) & \text{if} \;\; k_0(T) = 1 \end{cases}$$

and the lemma is therefore valid when $k_0(T) \leq 1$.

So suppose $k_0(T) = k \geq 2$, and that the lemma has been proved for any tangle with colour less than k. Since T has no internal discs, it must be the case that T contains a string which connects two adjacent marked points on ∂D_0. We see that there must be an integer $0 \leq p < k$ such that $T_1 = R_k^{(p)} \circ T$ contains a string which connects the points on ∂D_0 that are labelled either (i) k and $k+1$, or (ii) $k-1$ and k. In either case, we see from Lemma 3.9(a),(b) that T_1 is expressible as a composite involving only the generators of Theorem 3.5 and a $(k-1)$-tangle without any internal discs. (If that $(k-1)$-tangle had internal discs, so would T, contrary to the hypothesis.) So, by induction hypothesis, we see that $T_1 \in \mathcal{T}_0$, and hence, $T = R_k^{(k-p)} \circ T_1 \in \mathcal{T}_0$.

□

LEMMA 3.11 *If A is any 'annular tangle' - i.e., if A has exactly one internal disc - and if A has no loops, then $A \in \mathcal{T}_0$.*

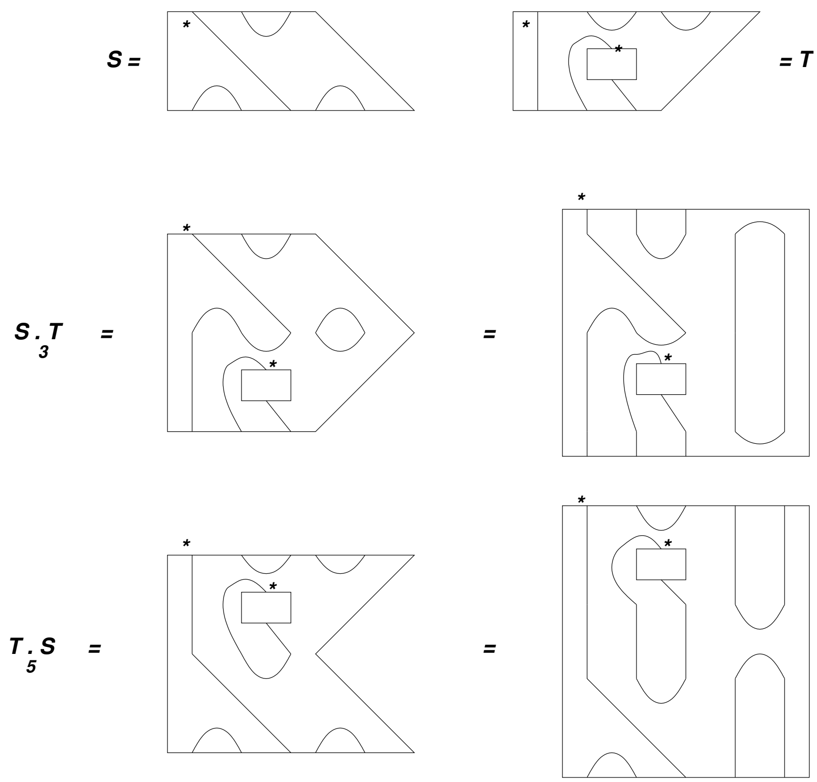
Proof: The following terminology will be helpful: let us agree to say that A has a *cap* if it contains a string which joins two labelled points on ∂D_i for some $0 \leq i \leq b(A)$, and to further refer to such a cap as *external* or *internal* according as $i = 0$ or $i > 0$.

As a first step, it should be clear that we may write $A = A_1 \circ A_2$ where neither A_1 nor A_2 has any loops, and A_1 has no internal cap and A_2 has no external cap. (For this, we only need to choose $\partial D_1(A_1) = \partial D_0(A_2)$ as (a diffeomorphic image of) a circle in the annular region of A which does not meet any cap of A.) So it suffices to prove that $A_i \in \mathcal{T}_0, i = 1, 2$.

Suppose A_i has m_i caps, for $i = 1, 2$. If $m_i = 0$, then A_i must[2] be a suitable 'power of the rotation' and so belongs to $\mathcal{T}_0$. If $m_i > 0$, then Lemma 3.9 allows us to express A_i as a composition involving rotations (to get to a situation where Lemma 3.9 is applicable), the permissible generators in $\mathcal{T}_0$, and one annular tangle A_i' of the same sort as A_i (only caps of one kind, internal or external, and no loops) but with one fewer number of caps than A_i; and an induction on the number m of caps completes the proof. □

We will find it convenient to introduce one final bit of notation: if k, l are two non-negative integers with $k+l = 2m$ for some non-negative integer m, and if S, T are m-tangles, then we shall write $S \cdot_k T$ to denote the k-tangle obtained by identifying the marked point labelled $2m - j + 1$ on $D_0(S)$ and the marked point labelled j on $D_0(T)$, for $1 \leq j \leq l$. We also demand that the internal discs of $S \cdot_k T$ are numbered by 'first listing those of S and then those of T'. The following illustrations, with $(i)k = 3, l = 5$ and $(ii)k = 5, l = 3$, should clarify the definition as well as the following lemma.

[2]The assumption about 'no loops' is necessary for us to be able to make this assertion.

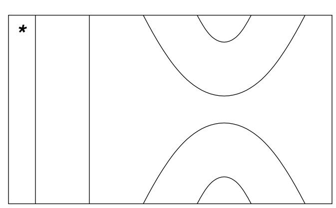


LEMMA 3.12 *If S, T are m-tangles, with $2m = k + l$ as above, we have*

$$S \cdot_k T = \begin{cases} E_m^k \circ (M_m \circ (S,T)) & if\ k \leq m \\ (M_k \circ_{D_1} (M_k \circ (L, I_m^k(T))) \circ_{D_2} I_m^k(S) & if\ k \geq m \end{cases},$$

where L is the 'Temperley-Lieb' k-tangle with no internal discs, and with strings joining the marked points labelled (a) r and $(2k - r + 1)$ for $1 \leq r \leq m$, (b) $(m + j)$ and $(k - j + 1)$ for $1 \leq j \leq \frac{k-m}{2}$, and (c) $(k + j)$ and $(2k - m - j + 1)$ for $1 \leq j \leq \frac{k-m}{2}$.

Proof: The proof only involoves staring at pictures such as the ones preceding the statement of the lemma. Perhaps we should mention that the 'Temperley-Lieb tangle L in the statement of the lemma consists of m strands 'coming straight down', and two sets of $\frac{k-m}{2}$ concentric caps, one set on the top, and one set at the bottom. We illustrate the case of $k = 6, m(= l) = 2$ below:



□

All the pieces are now in place for proving our result.

Proof of Theorem 3.5:

We shall first show that if T is any tangle without any loops, then $T \in \mathcal{T}_0$. We shall prove this by induction on the number $b(T)$ of internal discs of T. We have already proved the special cases $b(T) = 0$ and $b(T) = 1$ in Lemma 3.10 and Lemma 3.11, respectively. So assume $b(T) > 1$ and $k_0(T) = k$.

A moment's consideration shows that we may find an integer l of the same (even/odd-) parity as k and m-tangles T_1, T_2 - with $2m = k + l$, such that $T = T_1 \cdot_k T_2$, and further, $b(T_1) = 1$ and $b(T_2) = b(T) - 1$ (and of course, $D_1(T_1) = D_1(T)$ and $D_i(T_2) = D_{i+1}(T)$ for $i \geq 1$). The fact that T has no loops is seen to imply that the T_i's do not have any loops, either. Then $T_1 \in \mathcal{T}_0$ by Lemma 3.11, and $T_2 \in \mathcal{T}_0$ by the induction hypothesis; while it follows then from Lemma 3.12 that $T = T_1 \cdot_k T_2 \in \mathcal{T}_0$.

Thus we have indeed shown that if T is a tangle without any loops, then $T \in \mathcal{T}_0$. Then, it follows from Remark 3.7 that $\mathcal{T}_0$ must contain every tangle, and the proof is complete.

Finally this completes the proof of both Theorems 3.5 and 3.3. □

We now present an application of Theorem 3.5.

Recall that a subfactor planar algebra P is said to have depth at most d if $P_d e_d P_d = P_{d+1}$. By the AF C^*-algebra associated to P is meant the norm closure of $\cup_k P_k$, filtered by the P_k's. The content of the next proposition - when stated less concisely but more precisely - is that if P and Q are subfactor planar algebras with depth at most d, then the following conditions are equivalent:

(i) there exists C^*-algebra isomorphisms $\phi_k : P_k \to Q_k$, for $k \in \ Col$ such that

(a)

$$Z^Q_{I_k^{k+1}} \circ \phi_k = \phi_{k+1} \circ Z^P_{I_k^{k+1}} \ , \ \forall \ k \in \ Col$$

(b)

$$Z^Q_{R_{d+2}} \circ \phi_{d+2} = \phi_{d+2} \circ Z^P_{R_{d+2}} \ ;$$

and

(c)

$$Z^Q_{\mathcal{E}_k}(1) = \phi_{k+1}(Z^P_{\mathcal{E}_k}(1)) \ ;$$

(ii) the maps $\{\phi_k : k \in \ Col\}$ define an isomorphism of the subfactor planar algebras P and Q.

(We shall refer to the conditions (a) - (c) above by the statement that 'ϕ intertwines the actions of the tangles I_k^{k+1}, R_{d+2} and $\mathcal{E}_k$ respectively'.)

Thus, we have:

PROPOSITION 3.13 *A necessary and sufficient condition for two subfactor planar algebras P and Q with finite depth $\leq d$ to be isomorphic is that there exists an isomorphism of the associated AF C^*-algebras which (a) intertwines the actions of the tangle $Z_{R_{d+2}}$, and (b) maps the Jones projections to themselves.*

Proof: Let $\mathcal{C}$ denote the set of those tangles T such that 'ϕ intertwines the actions of T' - i.e., if T is a k_0-tangle with b internal discs with colours $k_1, \cdots, k_b$, then $T \in \mathcal{C}$ if

$$\phi_{k_0}(Z_T^P(x_1 \otimes \cdots \otimes x_b)) = Z_T^Q(\phi_{k_1}(x_1) \otimes \cdots \otimes \phi_{k_b}(x_b)) \forall\ x_i \in P_{k_i}$$

It follows easily from the definitions that $\mathcal{C}$ is closed under composition. Hence, in order to show that $\mathcal{C}$ contains all tangles - i.e., that the ϕ_k yield an isomorphism of subfactor planar algebras - it suffices, by Theorem 3.5, to verify that $\mathcal{C}$ contains all the 'generators' of that theorem.

Also, the finite-depth hypothesis implies that the AF algebras under discussion have a unique positive tracial state, which must necessarily be preserved by ϕ; it follows from the uniqueness of the trace-preserving conditional expectation that $\mathcal{C}$ must also contain the E_{k+1}^k.

The fact that the ϕ_k yield an isomorphism ϕ of the associated AF C^*-algebras implies that $\mathcal{C}$ contains $1_{0_\pm}$ (since ϕ is unital), the M_k (since ϕ preserves products) and the I_k^{k+1}'s (by assumption). Condition (i)(c) above can now be seen to imply, more generally, that

$$Z_T^Q(1) = \phi_{k+1}(Z_T^P(1))$$

for all 'Temperley-Lieb tangles.

Thus, we only need to verify that $R_k \in \mathcal{C}\ \forall k$, which will be done once we prove the assertions contained in the following three steps - during the course of whose proofs, we shall repeatedly use the fact that $\mathcal{C}$ is closed under composition and that $\mathcal{C}$ contains the M_k's, the I_k^{k+1}'s and the E_{k+1}^k's.

Step 1:

$$R_{k+1} \in \mathcal{C} \Rightarrow R_k \in \mathcal{C}$$

This is a consequence of the identity

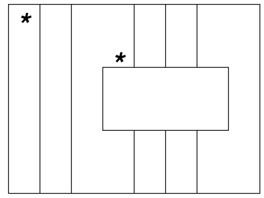
$$R_k = E_{k+1}^k \circ (M_{k+1} \circ (\mathcal{E}_k, R_k \circ I_{k-1}))$$

Step 2:

$$R_{k+2} \in \mathcal{C} \Rightarrow Sh_{k+2} \in \mathcal{C}\ ,$$

where Sh_{k+2} denotes the 'right shift by 2' which introduces two vertical lines to the left and 'shifts a k-box to the right by 2'. Thus, for instance, the tangle

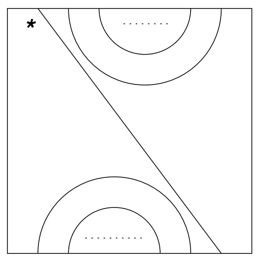
Sh_5 is given as follows:

$Sh_5 =$ 

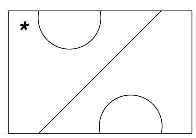
This is a consequence of the identity

$$Sh_{k+2} = \begin{cases} R_{k+2}^{(\frac{k+2}{2})} \circ (I_k^{k+2} \circ R_k^{(\frac{k}{2})}) & \text{if } k \textit{ is even} \\ R_{k+2}^{(k+1)} \circ (M_{k+2} \circ (I_k^{k+2}, L_{k+2})) & \text{if } k \textit{ is odd} \end{cases},$$

where L_{k+2} is the 'Temperley-Lieb tangle' (of colour $k+2$) defined by different prescriptions, for $k > 1$ and for $k = 1$, thus:

$L_k =$ 

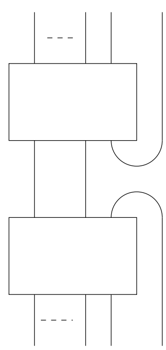
$(k > 1)$

$L_1 =$ 

Step 3:

$$R_k, Sh_{k+2} \in \mathcal{C}, k \geq d \Rightarrow R_{k+1}, Sh_{k+3} \in \mathcal{C}$$

Let H_{k+1} denote the $(k+1)$ tangle (with two internal discs of colour k) defined as follows:

$H_{k+1} =$ 

The assumption that $k \geq d$ implies that H_{k+1} is surjective, and so, in order to prove that a tangle T with a unique internal disc of colour $(k+1)$ belongs to $\mathcal{C}$, it will suffice to show that $T \circ H_{k+1} \in \mathcal{C}$.

The assertion regarding Sh_{k+3} now follows from the identity

$$Sh_{k+3} \circ H_{k+1} = M^{(3)}_{k+3}(I^{k+3}_{k+2} \circ Sh_{k+2}, \mathcal{E}_{k+3}, I^{k+3}_{k+2} \circ Sh_{k+2})$$

where we use the short-hand

$$M^{(3)}_m(T_1, T_2, T_3) = (M_m \circ_{D_2} T_3) \circ_{D_1} ((M_m \circ_{D_2} T_2) \circ_{D_1} T_1) \ .$$

We

4. New planar algebras from old

In this section, we want to consider a simple-minded method of constructing 'new planar algebras from old'. Our approach will be via *operations on tangles.* To be precise, we shall say we have an operation on tangles if we have self-maps $k \mapsto k^\#$ of the set Col of colours and $T \mapsto T^\#$ of the collection of tangles, subject to the following conditions:

(a) $b(T^\#) = b(T)$;

(b) $k_i(T^\#) = k_i(T)^\# \ \forall \ 0 \leq i \leq b(T)$; and

(c) if $T \circ_{D_i} S$ makes sense, then (so does $T^\# \circ_{D_i} S^\#$ and)

$$T^\# \circ_{D_i} S^\# \ = \ (T \circ_{D_i} S)^\# \ .$$

(d) $(I^k_k)^\# = I^{k^\#}_{k^\#} \ \forall \ k \in \ Col.$

In each of the following examples, it is not hard to see that our prescription does indeed yield an 'operation on tangles'.

EXAMPLE 4.14 *(a) (**Cabling**) Fix a positive integer m and define $k^\# = mk$, and for a tangle T, define $T^\#$ to be the result of 'blowing up each string of T by a factor of m'. We will not elaborate on this example as we will not be needing it here. We just wanted to mention it since such constructions are familiar in the theory, from the work of Jones, Wenzl, etc.*

*(b) (**Dual:**) Define*

$$k^- \ = \ \begin{cases} 0^\mp & if \ k = 0^\pm \\ k & if \ k \geq 1 \end{cases}$$

Given a tangle $T = (T, f)$ (as in the definition of a tangle), define $T^- = (T^-, f^-)$, where

(i) $T^- = T$ as a one-manifold, (but with the opposite orientation);

(ii) if $k_i(T) > 0$ for some i - so that $f(i)$ is defined - define $f^-(i)$ to be the first marked point that is met on proceeding in an anti-clockwise direction from $f(i)$ on $\partial D_i(T)$; (thus $f^-(i)$ is the point that was labelled $2k_i(T) - 1$;) and

(iii) the checkerboard shading on T^- is opposite to that of T.

The reason for terming this 'dual' will be seen shortly.

*(c) (**Flip:**) Define $k^\sim = k^-$ (as in (b) above), and let $T^\sim$ denote the tangle obtained by 'reflecting T'; more precisely, if ϕ is any orientation-reversing map of the plane, define the one-manifold underlying $T^\sim$ to be $\phi(T)$, set $D_i(T^\sim) = \phi(D_i(T))$ for $1 \leq i \leq b(T)$, choose the distinguished marked point on $\partial D_i(T^\sim)$ as the image under ϕ of the distinguished marked point on $\partial D_i(T)$, and choose the colour (black or white) of $\phi(R)$ - where R is a 'region' of one colour in T - to be opposite to that of the region R. A little thought shows that this 'flip' is indeed an operation on tangles with the desired properties.*

*(d) (**Adjoint:**) Define $k^* = k \ \forall \ k$ and let T^* denote the adjoint of the tangle T, as described earlier. It should be noted that that the adjoint is the composition of the dual and flip operations; i.e., $T^* = (T^-)^\sim$ for every tangle, or in other words, simply, $* = - \sim$.*

(e) The operations 'adjoint' and 'flip' turn out to be 'involutory', i.e., of order two, but they do not commute. The easiest - and most useful - way of seeing this is to note that 'dual' is not an involution on the collection of tangles; in fact, (writing $b(T) = b, k_i(T) = k_i, D_i(T) = D_i$), we see that the composition $(-)^2 = --$ is given by

$$(T^-)^- \ = \ (R_{k_0})^{(-1)} \circ (T \circ_{(D_1,\cdots,D_b)} (R_{k_1}, \cdots R_{k_b})) \ , \tag{4.8}$$

where we have used the obvious notation $(R_k)^{(-1)}$ to denote the 'inverse' annular tangle - which is also the $(k-1)$-fold iterated composition $(R_k)^{(k-1)}$.

Thus, under composition, the two involutions $*$ and $\sim$ may be expected to generate infinitely many different operations of tangles, but we shall see that from the point of view of planar algebras, this is not the case.

Given an operation $T \mapsto T^\#$ of tangles, we also get an 'operation on planar algebras' in the following manner. Thus if $P = \{P_k : k \in \ Col\}$ is a planar algebra, let us define another planar algebra ${}^\# P = \{({}^\# P)_k : k \in \ Col\}$ with the k-boxes given by

$$({}^\# P)_k = P_{k^\#}$$

and the multilinear mappings $Z_T^{{}^\# P}$ given by

$$Z_T^{{}^\# P} = Z_{T^\#}^P \ .$$

Note that this definition makes sense; for instance, the left and right sides of the above equation are maps $\otimes_{i=1}^{b(T)} ({}^\# P)_{k_i(T)} \to ({}^\# P)_{k_0(T)}$ and $\otimes_{i=1}^{b(T^\#)} P_{k_i(T^\#)} \to$

$P_{k_0(T^\#)}$ respectively; and these domains and co-domains agree by our requirements (a) and (b) for an operation of tangles. The fact that this specification satisfies the 'associativity requirement' for the assignment $T \mapsto Z_T$ is a consequence of our requirement (c), while we need the requirement (d) to ensure that ${}^\#P$ satisfies the 'non-degeneracy' condition to be fulfilled by all planar algebras. So ${}^\#P$ is indeed a planar algebra.

REMARK 4.15 *Our seemingly peculiar notation - of writing ${}^\#P$ and $T^\#$ - is motivated by 'functorial reasons'; only by allowing ourselves this 'left-right' flexibility can we ensure, for instance, that ${}^{-*}P = {}^-({}^*P)$; indeed,*

$$Z_T^{{}^{-*}P} \;=\; Z_{T^{-*}}^P \;=\; Z_{(T^-)^*}^P \;=\; Z_{T^-}^{{}^*P} \;=\; Z_T^{{}^-({}^*P)} \;.$$

The next result justifies our terminology of calling ${}^-P$ the 'dual' of the planar algebra P.

PROPOSITION 4.16 *For any planar algebra P, we have*

$${}^{--}P \cong P \;,$$

and consequently, also

$${}^{*-}P \cong {}^{\sim}P \cong {}^{-*}P \;.$$

Proof: Since $k^{--} = k$, we do have $({}^{--}P)_k = P_k \;\forall k$. Let us set R_k to be equal to I_k^k if $k < 2$ (since we had earlier defined R_k only for $k \geq 2$), and define $\pi_k = Z_{R_k}^P$ for all $k \in \; Col$. We shall show that $\pi = \{\pi_k : k \in \; Col\}$ defines an isomorphism of the planar algebra ${}^{--}P$ onto the planar algebra P.

Since clearly π_k is a linear isomorphism of $({}^{--}P)_k$ onto P_k for each k - both underlying vector spaces being just P_k -, we only need to check that if T is a tangle - as in Example 4.14(e), say - and if $x_i \in P_{k_i}$, then

$$Z_T^P(\otimes_{i=1}^b \pi_{k_i}(x_i)) \;=\; \pi_{k_0}(Z_T^{{}^{--}P}(\otimes_{i=1}^b x_i)) \;;$$

or in other words, that

$$Z_T^P(\otimes_{i=1}^b \pi_{k_i}(x_i)) \;=\; \pi_{k_0}(Z_{T^{--}}^P(\otimes_{i=1}^b x_i)) \;,$$

but this is exactly what is implied by equation (4.8).

As for the last assertion of the proposition, recall - from Example 4.14(d) - that $* = - \sim$, and consequently, $-* = -- \sim$ and so

$${}^{-*}P = {}^{--\sim}P = {}^{--}({}^{\sim}P) \cong {}^{\sim}P \;;$$

on the other hand, we have $*- = - \sim -$, and the definitions show that $T^{-\sim-} = T^{\sim}$ for any tangle T, and so, indeed

$${}^{*-}P = {}^{\sim}P \cong {}^{-*}P \;.$$

□

We now proceed to describe the relevance of the above examples in the context of subfactors.

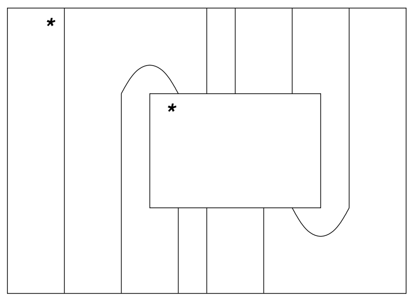
PROPOSITION 4.17 *Suppose $N \subset M$ is an extremal subfactor and $P = P^{N \subset M}$. Then,*
(a) $^{-}P \cong P^{M \subset M_1}$;
(b) $^{\sim}P \cong P^{M^{op} \subset M_1^{op}}$; and
(c) $^{*}P \cong P^{N^{op} \subset M^{op}}$

Proof: (a) Define

$$Q_k \; = \; \begin{cases} \mathbb{C} & if \;\; k = 0_{\pm} \\ M' \cap M_k & if \;\; k \geq 1 \end{cases} \quad ,$$

and consider the 'Fourier transform tangle' $F = F_k^{k+1}$ (with colour $(k+1)$ and one internal disc of colour k) defined, for $k = 0_{\pm}$ as $F_{0_{\pm}}^1 = I_{0_{\pm}}^1$, and for $k \geq 1$ as illustrated below:

$$F \;=\; F_k^{k+1} \;=$$

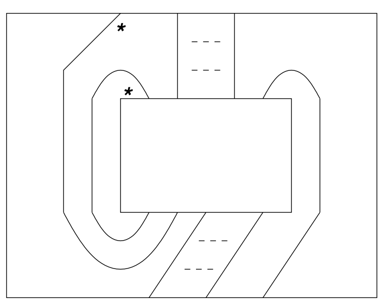


Then it is true - see [Jo] or [BiJo] - that $\phi_k = Z^P_{F_k^{k+1}}$ is a linear isomorphism of $(^{-}P_k =)\; P_k$ onto $Q_k \;(\subset\; P_{k+1})$. In fact, it is not hard to see that when the planar algebra has modulus δ, we have

$$Z_{IF^k_{k+1}} \circ Z_{F_k^{k+1}} = \delta \; id_{P_k} \;\; ,$$

where IF^k_{k+1} is the 'inverse Fourier transform tangle' given, for $k = 0_{\pm}$ as $IF_1^{0_{\pm}} = E_1^{0_{\pm}}$, and for $k \neq 0_{\pm}$ as illustrated below:

$$IF^k_{k+1} \;=$$



Let Q be the planar algebra obtained by 'transferring the planar algebra structure of ^-P via ϕ', i.e., by demanding that if T is a k-tangle, then

$$Z_T^Q(\otimes_i \phi_{k_i(T)} x_i) = \phi_{k_0(T)^-}(Z_T^{^-P}(\otimes_i x_i)) = \phi_{k_0(T)^-}(Z_{T^-}^P(\otimes_i x_i))$$

for all $x_i \in P_{k_i(T)}$.

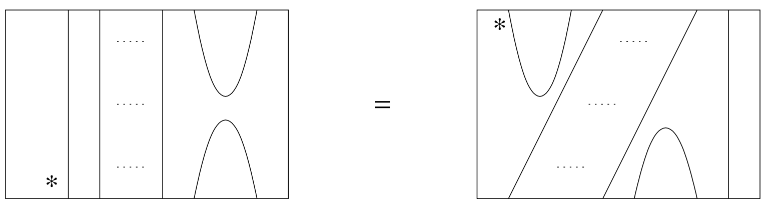
We need to prove that the planar algebra structure so defined on Q agrees exactly with the planar algebra structure on $P^{M\subset M_1}$; Since $Q_k = P_k^{M\subset M_1}$ for all $k \in \ Col$, we need to verify that

$$Z_T^Q \ = \ Z_T^{P^{M\subset M_1}} \tag{4.9}$$

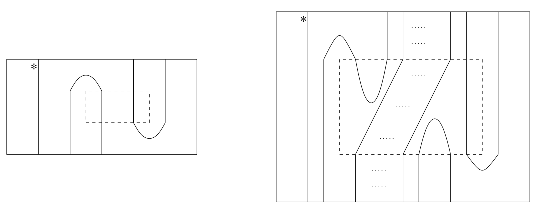
for every tangle T.

The proof of equation (4.9) will be by an application of Theorem 3.5; thus, setting $\mathcal{T}_0$ to be the collection of those tangles for which the conclusion of equation (4.9) is valid, we shall verify that $\mathcal{T}_0$ satisfies the hypotheses of Theorem 3.5, to arrive at the desired conclusion. We will only verify that $\{\mathcal{E}^k : k \geq 2\} \cup \{E_{k+1}^k : k \in \ Col\} \subset \mathcal{T}_0$. The other verifications are entirely similar.

Notice that $(\mathcal{E}_2)^- = I_2^2$ while if $k \geq 3$, $(\mathcal{E}_k)^-$ has precisely three sets of 'neighbours' which are connected, *viz.*. $(1,2)$, $(k, k+1)$ and $(k+2, k+3)$, as shown below:



Hence it is seen that $F_k^{k+1} \circ (\mathcal{E}_k)^- = \mathcal{E}^{k+1}$.



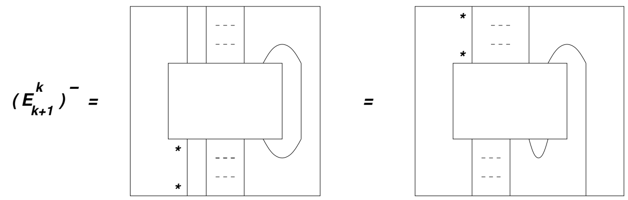
It follows from the definitions (and the fact that $k^- = k$ for $k \geq 1$) that

$$Z_{\mathcal{E}_k}^Q(1) = \phi_k \circ Z_{(\mathcal{E}_k)^-}^P(1) = Z_{F_k^{k+1}\circ \mathcal{E}_k^-}^P(1) = \delta \ e_k = Z_{\mathcal{E}_k}^{M\subset M_1}(1) \ ,$$

by two applications of Theorem 2.1(1).

We next consider the case $T = E_{k+1}^k$. Begin by observing that $(E_1^{0\pm})^- =$

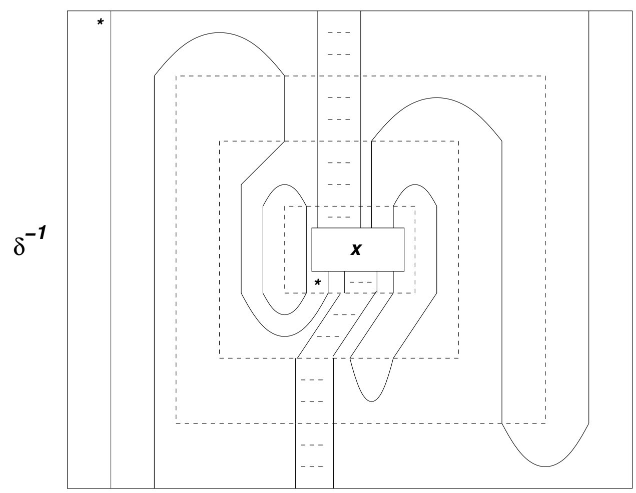
$E_1^{0\mp}$ and that for $k \geq 1$, $(E_{k+1}^k)^-$ is given as follows:



Now the definitions show that if $y = \phi_k(x) \in Q_{k+1}$ (with $x \in P_{k+1}$), then

$$\begin{aligned} Z^Q_{E^k_{k+1}}(y) &= \phi_{k^-} \circ Z^P_{(E^k_{k+1})^-}(x) \\ &= \delta^{-1} Z^P_{F^{k^-+1}_{k^-}} \circ Z^P_{(E^k_{k+1})^-} \circ Z^P_{IF^{k+1}_{k+2}}(y) \\ &= \delta^{-1} Z^P_{F^{k^-+1}_{k^-} \circ ((E^k_{k+1})^- \circ IF^{k+1}_{k+2})}(y)\ . \end{aligned}$$

On the other hand, an examination of the definitions shows that the right hand side of the above equation is seen to be given by:



Hence, we see that

$$\begin{aligned} Z^Q_{E^k_{k+1}}(y) &= Z^P_{E^{k+1}_{k+2}}(y) \\ &= \delta E_{N' \cap M_k}(y) \\ &= \delta E_{M' \cap M_k}(y) \quad (\textit{since } y \in M') \\ &= Z^{P^{M \subset M_1}}_{E^k_{k+1}}(y) \end{aligned}$$

as desired - where we have appealed twice to Theorem 2.1(3). (The cases $k = 0_\pm$ should obviously be treated slightly differently since $k^- \neq k$ in those cases; the case $k = 0_-$ needs the fact that P is spherical whence the 'left'- and 'right'-traces are the same.)

(c) It is fairly easy to see that there exists unital algebra anti-isomorphisms

$$M_n \ni x \overset{\gamma_n}{\mapsto} x^o \in M_n^{op}$$

so that the maps $M_n^{op} \to M_{n+1}^{op}$ defined by $\gamma_n(x) \mapsto \gamma_{n+1}(x)$ define inclusions (of II_1 factors) in such a way that

$$N^{op} \subset M^{op} \subset^{e_1^o} M_1^{op} \subset^{e_2^o} M_2^{op} \subset^{e_3^o} \cdots$$

is a tower of the basic construction.

On the other hand, we have (by definition) $({}^*P)_k = P_k$. Define

$$Q_k \;=\; \begin{cases} \mathbb{C} & if\ k = 0_\pm \\ (N^{op})' \cap M_k^{op} & if\ k \geq 1 \end{cases} ,$$

note that $\gamma_k(P_k) = Q_k$ and consider the maps $\pi_k : ({}^*P)_k \to Q_k$ defined by $\pi_k = \gamma_k|_{P_k}$. Since the π_k's are linear isomorphisms, we may, as in (a), define a planar algebra structure on Q by transporting the structure on ${}^*P$ via $\pi = \{\pi_k : k \in \ Col\}$; in other words, if T is any tangle - with $b = b(T), k_i = k_i(T)$, etc. - and if $x_i \in P_{k_i}$, we define

$$Z_T^Q(\otimes x_i^o) = \left(Z_T^{({}^*P)}(\otimes x_i)\right)^o = \left(Z_{T^*}^P(\otimes x_i)\right)^o \ .$$

What we need to show is that $Q = P^{N^{op} \subset M^{op}}$ as planar algebras. Since the two sides have the same underlying 'k-boxes', we only need to check that $Z_T^Q = Z_T^{P^{N^{op} \subset M^{op}}}$ for all tangles T. As in (a), we need to verify that this identity holds for all our generators.

The verification is easy. It starts by observing that all the generators in Theorem 3.5 - with the solitary exception of $M_k, k \geq 1$ - are 'self-adjoint'; as for M_k, the only difference between M_k and M_k^* is that the 'internal discs are reversed' so that, for instance, if $x_i \in ({}^*P)_k,\ i = 1, 2$, we have

$$\begin{aligned} Z_{M_k}^Q(x_1^o \otimes x_2^o) &= \left(Z_{M_k^*}^P(x_1 \otimes x_2)\right)^o \\ &= \left(Z_{M_k}^P(x_2 \otimes x_1)\right)^o \\ &= (x_2 x_1)^o \\ &= x_1^o x_2^o \\ &= Z_{M_k}^{P^{N^{op} \subset M^{op}}}(x_1^o \otimes x_2^o) \ . \end{aligned}$$

The fact that $Z_T^Q = Z_T^{P^{N^{op} \subset M^{op}}}$ for all the other generators T is a consequence of the properties of the map $x \mapsto x^o$ mentioned in the first paragraph of this proof of (c). For instance, for $T = E^k_{k+1}$, and $x \in P_{k+1}$, we have

$$\begin{aligned} Z^Q_{E^k_{k+1}}(x^o) &= Z^P_{E^k_{k+1}}(x)^o \\ &= \delta E_{N' \cap M_{k-1}}(x)^o \\ &= \delta E_{N^{op\prime} \cap M^{op}_{k-1}}(x^o) \\ &= Z^{P^{N^{op} \subset M^{op}}}_{E^k_{k+1}}(x^o)\ . \end{aligned}$$

(b) Indeed, if $P = P^{N \subset M}$, then ${}^-P \cong P^{M \subset M_1}$ (by (a)), so ${}^{*-}P \cong P^{M^{op} \subset M_1^{op}}$ (by (c)), and finally ${}^\sim P \cong {}^{*-}P$ (by the last assertion of Proposition 4.16). □

(We should mention that the description - in Proposition 4.17(a) - of the dual planar algebra can also be found in the latest version of [J1]. Jones' initial description of the dual planar algebra was a good deal more complicated, and our desire to simplify that description was one of the reasons for our embarking on this study; he seems to have inserted this description in one of the numerous updates that that preprint has undergone; and we only became aware of it recently.)

Remark 4.18 *As is well-known - see [O], [Sa], [Sz] - there is a close connection between Kac algebras and subfactors. In fact, every finite-dimensional Kac algebra H 'admits an outer action on the hyperfinite II_1 factor R' and the associated fixed-point subalgebra $R^H \subset R$ is the prototypical 'irreducible depth-two subfactor'. The subfactor planar algebra $P(H)$ associated to this subfactor has been described via generators and relations in [KLS]. (That description provides an alternative proof of the fact that the isomorphism class of the subfactor depends only on H and not on the action, and that H can be recovered from it.)*

The analysis of this section can be thought of as explaining why each Kac algebra H gives rise to exactly 3 other (closely related) Kac algebras, viz., H^{op}, H^* *and H^{*op} (and why various tricks one can try with 'op's and 'cop's and combinations thereof lead to one of these four Kac algebras). In fact, by using the results of [KLS], for instance, in combination with Proposition 4.17, one can show that if $N = R^H \subset R = M$, then*

$$\begin{aligned} M \subset M_1 &\cong R^{H^*} \subset R \\ N^{op} \subset M^{op} &\cong R^{H^{op}} \subset R \\ M^{op} \subset M_1^{op} &\cong R^{H^{*op}} \subset R\ . \end{aligned}$$

And our analysis shows that at the level of planar algebras, the operations $-$ and $\sim$ 'generate a copy of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$', so that a given planar algebra gives rise, via these operations, to at most the four planar algebras $P, {}^-P, {}^*P$ and ${}^\sim P$.

Finally, we can free ourselves of the C^*-requirement and extend these comments to cover the case of semisimple Hopf algebras over $\mathbb{C}$. This is because our Theorem 3.5 applies equally well to 'general' planar algebras which need not *be subfactor planar algebras, and because the analysis of [KLS] shows how to associate such planar algebras, call them $P(H)$, to semisimple Hopf algebras H, and because the planar algebra analogues of the above three equations can be shown to continue to hold, meaning that*

$$\begin{aligned} {}^{-}P(H) &\cong P(H^*) \\ {}^{*}P(H) &\cong P(H^{op}) \\ {}^{\sim}P(H) &\cong P(H^{*op}) \ . \end{aligned}$$

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