Quantum isometry groups: a brief overview

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Introduction

- Two important branches of ‘noncommutative mathematics’ : **Quantum groups** a la Woronowicz, Drinfeld, Jimbo and others, and **Noncommutative Geometry** a la Connes.
- However, a successful marriage of the two is still not fully achieved...some initial and pioneering success in this direction came from Chakraborty-Pal who constructed $SU_\mu(2)$-equivariant nontrivial spectral triple (which was later analyzed in depth by Connes).
- Since then, more examples of spectral triples which are equivariant w.r.t. quantum group actions have been constructed and studied by a number of mathematicians (Chakraborty, Pal, Landi, Dabrwoski, Sitarz, Hajac...just to mention a few).
- One may turn around and ask the question: given a spectral triple, what are all compact quantum group action on the underlying C^* algebra for which the given spectral triple is equivariant? This leads to the notion of **quantum isometry group**, which (if exists) should be the ‘biggest’ such quantum group.

- As in classical geometry, quantum isometry groups should play an important role in understanding noncommutative Riemannian geometry and more generally, noncommutative (quantum) metric space in the sense of Rieffel.
- Study of such quantum groups may also enrich quantum group theory.

We shall present a brief sketch of development of the theory of quantum isometry groups which is an outcome of collaboration with J. Bhowmick (JB), A. Skalski (AS) and T Banica (TB). This will include formulation in various frameworks, e.g. in terms of Laplacian (DG-CMP), formulation in terms of Dirac operator (JB+DG-JFA), for real spectral triples (DG-SIGMA), and also results about deformation (JB+DG-JFA), and computations for AF algebras (JB+DG+AS- Trans. AMS) Podles spheres (JB+DG- JFA), free and half liberated spheres (TB+DG- CMP) etc.

Quick review of basic concepts

Definition

a **compact quantum group** *(CQG for short) a la Woronowicz is a pair* $(\mathcal{A}, \Delta)$ *where* $\mathcal{A}$ *is a unital separable* C^**-algebra,* Δ *is a coassociative comultiplication, i.e. a unital* C^**-homomorphism from* $\mathcal{A}$ *to* $\mathcal{A} \otimes \mathcal{A}$ *(minimal tensor product) satisfying* $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$*, and each of the sets* $\{(b \otimes 1)\Delta(c) : b, c \in \mathcal{A}\}$ *and* $\{(1 \otimes b)\Delta(c) : b, c \in \mathcal{A}\}$ *generate dense linear subspaces of* $\mathcal{A} \otimes \mathcal{A}$

There is a natural generalisation of group action on spaces in this noncommutative set-up, which is given below :

Definition

We say that a CQG $(\mathcal{A}, \Delta)$ *acts on a (unital)* C^**-algebra* $\mathcal{C}$ *is there is a unital* $*$*-homomorphism* $\alpha : \mathcal{C} \to \mathcal{C} \otimes \mathcal{A}$ *such that* $(\alpha \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \Delta) \circ \alpha$*, and the linear span of* $\alpha(\mathcal{C})(1 \otimes \mathcal{A})$ *is norm-dense in* $\mathcal{C} \otimes \mathcal{A}$.

Noncommutative geometry a la Connes

Definition

A **spectral triple** *or* **spectral data** *is a tuple* $(\mathcal{A}, \mathcal{H}, D)$ *where* $\mathcal{H}$ *is a separable Hilbert space,* $\mathcal{A}$ *is a* $*$*-subalgebra of* $\mathcal{B}(\mathcal{H})$ *(not necessarily norm-closed) and* D *is a self-adjoint (typically unbounded) operator such that for each* $a \in \mathcal{A}$, *the operator* $[D, a]$ *admits bounded extension. Such a spectral triple is also called an* **odd spectral triple**. *If in addition, we have* $\gamma \in \mathcal{B}(\mathcal{H})$ *satisfying* $\gamma = \gamma^* = \gamma^{-1}$, $D\gamma = -\gamma D$ *and* $[a, \gamma] = 0$ *for all* $a \in \mathcal{A}$, *then we say that the quadruplet* $(\mathcal{A}, \mathcal{H}, D, \gamma)$ *is an* **even spectral triple** *or* **even spectral data**. *The operator* D *is called the* **Dirac operator** *corresponding to the spectral triple.*
We say that the spectral triple is of **compact type** *if* D *has compact resolvents. It is* Θ*-summable if* $Tr(e^{-tD^2}) < \infty$ *for* $t > 0$.

The motivation of this formulation comes from the typical classical examples of spectral triple associated with a Riemannian spin manifold M, where $\mathcal{H}$ can be chosen as the Hilbert space of square integrable sections of the spinor bundle, D as the Dirac operator, and $\mathcal{A}$ as $C^\infty(M)$ acting by multiplication on the sections of spinor bundle. In this case, the spectral triple contains full information about the underlying topology and the Riemannian metric.

Background

- Early work : formulation of quantum automorphism and quantum permutation groups by Wang, following suggestions of Alain Connes, and follow-up work by Banica, Bichon and others.
- Basic principle: For some given mathematical structure (e.g., a finite set, a graph, a C^* or von Neumann algebra) identify (if possible) the group of automorphisms of the structure as a universal object in a suitable category, and then, try to look for the universal object in a similar but bigger category by replacing groups by quantum groups of appropriate type.
- However, most of the earlier work done concerned some kind of quantum automorphism groups of a 'finite' structure. So, one should extend these to the 'continuous'/ 'geometric' set-up. This motivated my definition of quantum isometry group in [5].

Wang's quantum permutation and quantum automorphism groups

Quantum permutation group (Wang):
Let $X = \{1, 2, ..., n\}$, G group of permutations of X. G can be identified as the universal object in the category of groups acting on X. For a simiar (bigger) category of compact quantum groups acting on $C(X)$, Wang obtained the following universal object:

$$\mathcal{Q} := C^* \left(q_{ij}, i, j = 1, ..., n; \mid q_{ij} = q_{ij}^* = q_{ij}^2, \sum_i q_{ij} = 1 = \sum_j q_{ij} \right).$$

The co product is given by $\Delta(q_{ij}) = \sum_k q_{ik} \otimes q_{kj}$, and the action on $C(X)$ is given by $\alpha(\chi_i) = \sum_j \chi_j \otimes q_{ji}$.
This CQG is naturally called 'quantum permutation group' of n objects.

A similar question can be asked for finite dimensional matrix algebras. However, the answer is negative, i.e. the cateogory of CQG acting on M_n does NOT have a universal object!
Remedy (due to Wang): consider the subcategory of actions which preserves a given faithful state.
More precisely: For an $n \times n$ positive invertible matrix $Q = (Q_{ij})$, let $A_u(Q)$ be the universal C^*-algebra generated by $\{u_{kj}, k, j = 1, ..., d_i\}$ such that $u := ((u_{kj}))$ satisfies

$$uu^* = I_n = u^* u, \quad u' Q \overline{u} Q^{-1} = I_n = Q \overline{u} Q^{-1} u'.$$

Here $u' = ((u_{ji}))$ and $\overline{u} = ((u_{ij}^*))$. This is made into a CQG by the coproduct given by $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$.

Proposition

$A_u(Q)$ is the universal object in the category of compact quantum groups which admit an action on the finite dimensional C^* algebra $M_n(\mathbb{C})$ which preserves the functional $M_n \ni x \mapsto \mathrm{Tr}(\mathrm{Q}^{\mathrm{T}}\mathrm{x})$.

Quantum isometry in terms of ‘Laplacian’ (Goswami 2009)

- Classical ismoetries: the group of Riemannian isometries of a compact Riemannian manifold M is the universal object in the category of all compact metrizable groups acting on M, with smooth and isometric action.
- Moreover, a smooth map γ on M is a Riemannian isometry if and only if the induced map $f \mapsto f \circ \gamma$ on $C^\infty(M)$ commutes with the Laplacian $-d^*d$.

Under reasonable regularity conditions on a (compact type, Θ-summable) spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$, one has analogues of Hilbert space of forms $\mathcal{H}_i^D$, say, $i = 0, 1, \ldots$. The map $d(a) := [D, a]$ then extends to a (closable, densely defined) map from $\mathcal{H}_0^D$ (space of 0-forms) to $\mathcal{H}_1^D$ (space of one-forms). The self-adjoint negative map $-d^*d$ is the noncommutative analogue of Laplacian $\mathcal{L} \equiv \mathcal{L}_D$, and we additionally assume that
(a) $\mathcal{L}$ maps $\mathcal{A}^\infty$ into itself;
(b) $\mathcal{L}$ has compact resolvents and its eigenvectors belong to $\mathcal{A}^\infty$ and form a norm-total subset of $\mathcal{A}$;
(c) the kernel of $\mathcal{L}$ is one dimensional (“connectedness”).

It is then natural to call an action α of some CQG $\mathcal{Q}$ on the C^*-completion of $\mathcal{A}^\infty$ to be *smooth and isometric* if for every bounded linear functional ϕ on $\mathcal{Q}$, one has $(\mathrm{id} \otimes \phi) \circ \alpha$ maps $\mathcal{A}^\infty$ into itself and commutes with $\mathcal{L}_D$.

Theorem

Under assumptions (a)-(c), there exists a universal object (denoted by $QISO^{\mathcal{L}}$) in the category of CQG acting smoothly and isometrically on the given spectral triple.

- The assumption (c) can be relaxed for classical spectral triples and their Rieffel-deformations, i.e. the above existence theorem applies to arbitrary compact Riemannian manifolds (not necessarily connected) and their Rieffel deformations.

Quantum isometry in terms of the Dirac operator (Bhowmick-Goswami 2009)

- From the NCG perspective, it is more appropriate to have a formulation in terms of the Dirac operator directly.
- Classical fact: an action by a compact group G on a Riemannian spin manifold is an orientation-preserving isometry if and only if lifts to a unitary representation of a 2-covering group of G on the Hilbert space of square integrable spinors which commutes with the Dirac operator.

For a spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ of compact type, it is thus reasonable to consider a category $\mathbf{Q}'$ of CQG $(\mathcal{Q}, \Delta)$ having unitary (co-) representation, say U, on $\mathcal{H}$, (i.e. U is a unitary in $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{Q})$ such that $(\mathrm{id} \otimes \Delta)(U) = U_{12}U_{13}$) which commutes with $D \otimes 1_{\mathcal{Q}}$, and for every bounded functional ϕ on $\mathcal{Q}$, $(\mathrm{id} \otimes \phi) \circ \mathrm{ad}_U$ maps $\mathcal{A}^\infty$ into its weak closure. Objects of this category will be called 'orientation preserving quantum isometries'.

- If $\mathbf{Q}'$ has a universal object, we denote it by $\widetilde{QISO^+}(D)$. In general, however, $\mathbf{Q}'$ may fail to have a universal object.
- We discussed in [3] some sufficient conditions, such as the existence of a suitable cyclic separating eigenvector of D, to ensure that a universal object exists in $\mathbf{Q}'$.

In general, we do get a universal object in suitable subcategories by fixing a 'volume form'...

Theorem

Let R be a positive, possibly unbounded, operator on $\mathcal{H}$ commuting with D and consider the functional (defined on a weakly dense domain) $\tau_R(x) = Tr(Rx)$. Then there is a universal object (denoted by $\widetilde{QISO^+}_R(D)$) in the subcategory of $\mathbf{Q}'$ consisting of those $(\mathcal{Q}, \Delta, U)$, for which $(\tau_R \otimes \mathrm{id})(\mathrm{ad}_U(\cdot)) = \tau_R(\cdot)1_{\mathcal{Q}}$.

Given such a choice of R, we shall call the spectral triple to be R-twisted.

- The C^*-subalgebra $QISO_R^+(D)$ of $\widetilde{QISO}_R^+(D)$ generated by elements of the form $\{< (\xi \otimes 1), \mathrm{ad}_{U_0}(a)(\eta \otimes 1) >, \ a \in \mathcal{A}^\infty\}$, where U_0 is the unitary representation of $\widetilde{QISO^+}_R(D)$ on $\mathcal{H}$ and $< \cdot, \cdot >$ denotes the $\widetilde{QISO}_R^+(D)$-valued inner product of the Hilbert module $\mathcal{H} \otimes \widetilde{QISO}_R^+(D)$, will be called the quantum group of orientation and (R-twisted) volume preserving isometries. A similar C^*-subalgebra of $\widetilde{QISO}^+(D)$, if it exists, will be denoted by $QISO^+(D)$.
- However, $QISO_R^+(D)$ may not have C^* action for noncommutative manifolds, and the subcatogory of $\mathbf{Q}'_R$ with those objects for which ad_U gives a C^* action, or even it maps into the C^* algebra, does not in general admit a universal object.
- Under mild conditions (valid for classical manifolds and their deformations), $QISO^{\mathcal{L}}$ coincides with $QISO_I^+(d + d^*)$, where $d + d^*$ is the 'Hodge Dirac operator' on the space of all (noncommutative) forms.

Some sufficient conditions for C^* action

However, one can prove that ad_U gives a C^* action of $QISO_R^+$ for a rather large class of spectral triples, including the cases mentioned below.

- For any spectral triple for which there is a 'reasonable' Laplacian in the sense of [5]. This includes all classical spectral triples as well as their Rieffel deformation (with $R = I$).
- Under the assumption that there is an eigenvalue of D with a one-dimensional eigenspace spanned by a cyclic separating vector ξ such that any eigenvector of D belongs to the span of $\mathcal{A}^\infty \xi$ and $\{a \in \mathcal{A}^\infty : \ a\xi \text{ is an eigenvector of } D\}$ is norm-dense in $\mathcal{A}$ (in this case the universal object in $\mathbf{Q}'$ exists).
- Under some analogue of the classical Sobolev conditions with respect to a suitable group action on $\mathcal{A}$.

QISO for spectral triples with real structures (Goswami 2010)

We have been able to prove existence of a quantum isometry group without fixing a 'volume form', provided we are given an additional structure, namely a real structure, which we describe below.

- A real structure for an odd spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ is given by a (possibly unbounded, invertible) closed anti-linear operator $\tilde{J}$ on $\mathcal{H}$ such that $\mathcal{D} := \mathrm{Dom}(D) \subseteq \mathrm{Dom}(\tilde{J})$, $\tilde{J}\mathcal{D} \subseteq \mathcal{D}$, $\tilde{J}$ commutes with D on $\mathcal{D}$, and the antilinear isometry J obtained from the polar decomposition of $\tilde{J}$ satisfies $J^2 = \epsilon I$, $JD = \epsilon' DJ$, and for all $x, y \in \mathcal{A}^\infty$, the commutators $[x, JyJ^{-1}]$ and $[JxJ^{-1}, [D, y]]$ are compact operators. Here, ϵ, ϵ' are ± 1, obeying the sign-convention described, e.g. in "An Introduction to Noncommutative Geometry", by J. C. Varilly (European Math. Soc., 2006).
- For the even case, additional requirement is some commutation relation of the form $J\gamma = \epsilon'' \gamma J$ for some $\epsilon'' = \pm 1$ between J and the grading operator γ.

Let $\mathbf{Q}'_{\rm real}$ be the subcategory of $\mathbf{Q}'$ of orientation preserving quantum isometries consisting of those $(\mathcal{Q}, U)$ for which following holds on $\mathcal{D}_0$ (the linear span of eigenvectors of D):

$$(\tilde{J} \otimes \tilde{J}_{\mathcal{Q}}) \circ U = U \circ \tilde{J}.$$

, where $\tilde{J}_{\mathcal{Q}}$ denotes the antilinear map sending x to x^*, for x belonging to the Hopf $*$-algebra $\mathcal{Q}_0$ generated by matrix elements of irreducible repreentations of $\mathcal{Q}$. Note that commutation of U and D implies that U must map $\mathcal{D}_0$ into the algebraic tensor product of $\mathcal{D}_0$ and $\mathcal{Q}_0$.

Theorem

The category $\mathbf{Q}'_{\rm real}$ admits a universal object, denoted by $\widetilde{QISO}_{\rm real}(D)$, and the C^*-subalgebra $QISO_{\rm real}(D)$ of $\widetilde{QISO}_{\rm real}(D)$ generated by elements of the form $\{< (\xi \otimes 1), {\rm ad}_{U_0}(a)(\eta \otimes 1) >, \ a \in \mathcal{A}^\infty\}$, where U_0 is the unitary representation of $\widetilde{QISO}_{\rm real}(D)$ on $\mathcal{H}$ is called the quantum group of orientation and real structure preserving isometries.

QISO of deformed noncommutative manifolds (Bhowmick-Goswami 2009)

Recall Rieffel deformation of C^* algebras and Rieffel-Wang deformation of CQG. we give a general scheme for computing quantum isometry groups by proving that $\widetilde{QISO_R^+}$ of a deformed noncommutative manifold coincides with (under reasonable assumptions) a similar (Rieffel-Wang) deformation of the $\widetilde{QISO_R^+}$ of the original manifold.

Let $(\mathcal{A}, \mathbb{T}^n, \beta)$ be a C^* dynamical system, $\mathcal{A}^\infty$ be the algebra of smooth (C^∞) elements for the action β., and D be a self-adjoint operator on $\mathcal{H}$ such that $(\mathcal{A}^\infty, \mathcal{H}, D)$ is an R-twisted, θ-summable spectral triple of compact type. Assume that there exists a compact abelian group $\widetilde{\mathbb{T}^n}$ with a covering map $\gamma : \widetilde{\mathbb{T}^n} \to \mathbb{T}^n$, and a strongly continuous unitary representation $V_{\tilde{g}}$ of $\widetilde{\mathbb{T}^n}$ on $\mathcal{H}$ such that

$$V_{\tilde{g}} D = D V_{\tilde{g}}, \;\; V_{\tilde{g}} a V_{\tilde{g}}^{-1} = \beta_g(a), \;\; g = \gamma(\tilde{g}).$$

Theorem

(i) For each skew symmetric $n \times n$ real matrix J, there is a natural representation of the Rieffel-deformed C^* algebra $\mathcal{A}_J$ in $\mathcal{H}$, and $(\mathcal{A}_J^\infty = (\mathcal{A}^\infty)_J, \mathcal{H}, D)$ is an R-twisted spctral triple of compact type.
(ii) If $QISO_R^+(\mathcal{A}_J^\infty, \mathcal{H}, D)$ and $(QISO_R^+(\mathcal{A}^\infty, \mathcal{H}, D))_{\tilde{J}}$ have C^* actions on $\mathcal{A}$ and $\mathcal{A}_J$ respectively, where $\tilde{J} = J \oplus (-J)$, we have

$$QISO_R^+(\mathcal{A}_J^\infty, \mathcal{H}, D) \cong (QISO_R^+(\mathcal{A}^\infty, \mathcal{H}, D))_{\tilde{J}}.$$

(iii) A similar conclusion holds for $QISO^+(\mathcal{A}^\infty)$, $QISO^+(\mathcal{A}_J^\infty)$ provided they exist.
(iv) In particular, for deformations of classical spectral triples, the C^* action hypothesis of (ii) or (iii) hold, and hence the above conclusions hold too.

- Similar deformation results hold for $QISO^{\mathcal{L}}$ under reasonable conditions, e.g. faithfulness of haar state of $QISO^{\mathcal{L}}$.

QISO for AF algebras (Bhowmick, Goswami, Skalski)

We have shown the $QISO^+$ is well-behaved w.r.t. the inductive limit construction, and used this principle to compute $QISO^+$ of many interesting spectral triples on AF algebras. A more precise result is the following:

Theorem

Suppose that $(\mathcal{A}^\infty, \mathcal{H}, D)$ is a spectral triple of compact type such that
(a) D has a one-dimensional eigenspace spanned by a vector ξ which is cyclic and separating for $\mathcal{A}$.
(b) There is an increasing sequence $(\mathcal{A}_n^\infty)_{n\in\mathbb{N}}$ of unital *-subalgebras of $\mathcal{A}^\infty$ whose union is $\mathcal{A}^\infty$, and D commutes with the projection P_n onto the closed subspace $\mathcal{H}_n$ generated by $\mathcal{A}_n^\infty \xi$ for each n.
Then each $(\mathcal{A}_n^\infty, \mathcal{H}_n, D_n := D|_{\mathcal{H}_n})$ is a spectral triple for which $QISO^+$ exists, and there exist natural compatible CQG morphisms $\pi_{m,n} : QISO^+(D_m) \to QISO^+(D_n)$, $m \leq n$ such that

$$QISO^+(D) = \lim_{n\in\mathbb{N}} QISO^+(D_n).$$

Examples and computations

Classical spaces

- for all the classical connected manifolds considered so far, such as the spheres and the tori, quantum isometry groups turn out to be the same the classical group of isometries, i.e. there are no genuine quantum isometries in these cases.
- For finite sets, our definition recovers the definitions given by Wang, Banica, Bichon and others.
- (JB+DG+AS): For the usual cantor set, the quantum isometry group turns out to be (as a C^* algebra) the universal C^*-algebra generated by the family of selfadjoint projections

$$\{p\} \cup \bigcup_{n \in \mathbb{N}} \{p_{m_1,\ldots m_n} : m_1, \ldots m_n \in \{1,2,3,4\}\}$$

subjected to the following relations:

$$p_1, p_2 \leq p, \quad p_3, p_4 \leq p^{\perp},$$

$$p_{m_1,\ldots,m_n,1}, p_{m_1,\ldots,m_n,2} \leq p_{m_1,\ldots,m_n}, \quad p_{m_1,\ldots,m_n,3}, p_{m_1,\ldots,m_n,4} \leq p^{\perp}_{m_1,\ldots,m_n}$$

$(n \in \mathbb{N}, m_1, \ldots m_n \in \{1,2,3,4\})$.

Noncommutative Tori (Bhowmick-Goswami 2009)

Consider the noncommutative two-torus $\mathcal{A}_\theta$ (θ irrational) generated by two unitaries U, V satisfying $UV = e^{2\pi i\theta} VU$, and the standard spectral triple on it described by Connes. Here, $\mathcal{A}^\infty$ is the unital $*$-algebra spanned by U, V; $\mathcal{H} = L^2(\tau) \oplus L^2(\tau)$ (where τ is the unique faithful trace on $\mathcal{A}_\theta$) and D is given by $D = \begin{pmatrix} 0 & d_1 + id_2 \\ d_1 - id_2 & 0 \end{pmatrix}$, where d_1 and d_2 are closed unbounded linear maps on $L^2(\tau)$ given by $d_1(U^m V^n) = mU^m V^n$, $d_2(U^m V^n) = nU^m V^n$. For this, there is a nice 'Laplacian' $\mathcal{L}$ given by $\mathcal{L}(U^m V^n) = (-m^2 + n^2)U^m V^n$.

Theorem

(i) $QISO^{\mathcal{L}} = \oplus_{k=1}^{8} C^*(U_{k1}, U_{k2})$ (as a C^* algebra), where for odd k, U_{k1}, U_{k2} are the two commuting unitary generators of $C(\mathbb{T}^2)$, and for even k, $U_{k1}U_{k2} = \exp(4\pi i\theta)U_{k2}U_{k1}$, i.e. they generate $\mathcal{A}_{2\theta}$.
(ii) $QISO^+(D)$ exists and coincides with $C(\mathbb{T}^2)$, i.e. there are no quantum orientation preserving isometries, although there are genuine quantum isometries in the Laplacian-based approach.

$U_\mu(2)$ as $QISO^+$ of $SU_\mu(2)$ (Bhowmick-Goswami 2009)

- The CQG $SU_\mu(2)$ $\mu \in [-1,1]$ is the universal unital C^* algebra generated by α, γ satisfying: $\alpha^*\alpha + \gamma^*\gamma = 1$, $\alpha\alpha^* + \mu^2\gamma\gamma^* = 1$, $\gamma\gamma^* = \gamma^*\gamma$, $\mu\gamma\alpha = \alpha\gamma$, $\mu\gamma^*\alpha = \alpha\gamma^*$., and the coproduct given by : $\Delta(\alpha) = \alpha \otimes \alpha - \mu\gamma^* \otimes \gamma$, $\Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma$.
- On the Hilbert space $L^2(h)$ (h Haar state), Chakraborty-Pal described a natural spectral triple with the D given by $D(e_{ij}^{(n)}) = (2n+1)e_{ij}^{(n)}$ if $n \neq i$, and $-(2n+1)e_{ij}^{(n)}$ for $n = i$, where $e_{ij}^{(n)}$ are normalised matrix elements of the $2n+1$ dimensional irreducible representation, n being half-integers.

Theorem

$QISO^+(D)$ is the CQG $U_\mu(2)$ which is the universal C^* algebra generated by $u_{11}, u_{12}, u_{21}, u_{22}$ satisfying: $u_{11}u_{12} = \mu u_{12}u_{11}$, $u_{11}u_{21} = \mu u_{21}u_{11}$, $u_{12}u_{22} = \mu u_{22}u_{12}$, $u_{21}u_{22} = \mu u_{22}u_{21}$, $u_{12}u_{21} = u_{21}u_{12}$, $u_{11}u_{22} - u_{22}u_{11} = (\mu - \mu^{-1})u_{12}u_{21}$ *and* $\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$ *is a unitary.*

Podles spheres (Bhowmick-Goswami 2010)

- The Podles sphere $S^2_{\mu,c}$ is the universal C^* algebra generated by A, B satisfying

$$AB = \mu^{-2}BA, A = A^* = B^*B + A^2 - cI = \mu^{-2}BB^* + \mu^2A^2 - c\mu^{-2}I.$$

- $S^2_{\mu,c}$ can also be identified as a suitable C^* subalgebra of $SU_\mu(2)$ and leaves invariant the subspace $\mathcal{K} = \overline{\mathrm{Span}\{e^{(l)}_{\pm\frac{1}{2},\ m} : l = \frac{1}{2}, \frac{3}{2}, ...,\ m = -l, -l+1, ...l\}}$ of $L^2(SU_\mu(2), h)$.
- R-twisted spectral triple given by: $D(e^{(l)}_{\pm\frac{1}{2},\ m}) = (c_1 l + c_2)e^{(l)}_{\mp\frac{1}{2},\ m}$, (where $c_1, c_2 \in \mathbb{R},\ c_1 \neq 0$), $R(e^{(n)}_{\pm\frac{1}{2},\ i}) = \mu^{-2i}e^{(n)}_{\pm\frac{1}{2},\ i}$.
- $$QISO^+_R(D) = SO_\mu(3) \equiv C^*\left(e^{(1)}_{ij},\ i,j = -1, 0, 1\right).$$
- There is also a real structure on this noncommutative manifold for which $QISO_{\mathrm{real}}$ turns out to be $SO_\mu(3)$.

Free and half liberated spheres (Banica-Goswami)

- Free sphere: $A_n^+ = C^* \left(x_1, \ldots, x_n \middle| x_i = x_i^*, \sum x_i^2 = 1 \right)$.
- It has a faithful trace, and in the corresponding GNS space we can construct a spectral triple for which the quantum isometry group is the free orthogonal group

$$O_n^+ = C^* \left(u_{11}, \ldots, u_{nn} \middle| u_{ij} = u_{ij}^*, u^t = u^{-1} \right).$$

- Similarly, consider the half-liberated sphere: $A_n^* = C^* \left(x_1, \ldots, x_n \middle| x_i = x_i^*, x_i x_j x_k = x_k x_j x_i, \sum x_i^2 = 1 \right)$.
- Again, for a natural spectral triple on this, we get the following CQG (half-liberated quantum orthogonal group) as the quantum isometry group:

$$O_n^* = C^* \left(u_{11}, \ldots, u_{nn} \middle| u_{ij} = u_{ij}^*, u_{ij} u_{kl} u_{st} = u_{st} u_{kl} u_{ij}, u^t = u^{-1} \right).$$

Open problems to be investigated

- Proving some general results about the structure and representation theory of such quantum isometry groups.
- Giving satisfactory sufficient conditions on the spectral triple to ensure the existence of the 'unrestricted' quantum isometry group $QISO^+(D)$
- Can there be a faithful action of a genuine (i.e. noncommutative as a C^* algebra) CQG on $C(X)$ for a connected compact space? Can such a CQG occur as a quantum isometry group of classical connected manifold?
- Extending the formulation of quantum isometry groups to the set-up of possibly noncompact manifolds (both classical and noncommutative), where one has to work in the category of locally compact quantum groups.
- Formulating a definition (and proving existence) of a quantum group of isometry for compact metric spaces , and more generally, for quantum metric spaces in the sense of Rieffel. Some wrok in this direction is done by Sabbe and Quaegebeur recently.

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