Noncommutative independence from the infinite Braid Group and Symmetric Group

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This talk

This talk is an introduction to the following two papers:

- R. Gohm & C. Köstler, *Noncommutative independence from the braid group $\mathbb{B}_\infty$.* Commun. Math. Phys. **282**, 435-482 (2009). (electronic: `arXiv:0806.3691v2`)
- R. Gohm & C. Köstler, *Noncommutative independence from characters of the symmetric group $\mathbb{S}_\infty$.* Preprint, 47 pages (2010). (electronic: `arXiv:1005.5726v1`)

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I refer to these papers as [GK-B] and [GK-S].

We begin with a short repetition of relevant material which has already been presented in more detail in Claus' talk. By [K] I refer to

- C. Köstler, *A noncommutative extended de Finetti theorem.* J. Funct. Anal. **258**, 1073–1120 (2010). (electronic: `arXiv:0806.3621v1`)

Thanks also to Claus for giving me some of his slides!

Distributional symmetries

Definition (Distributional symmetries)

A sequence of random variables $(\iota_n)_{n\in\mathbb{N}_0}$ from $(\mathcal{A}_0, \varphi_0)$ to $(\mathcal{A}, \varphi)$ is

(i) **exchangeable** if joint distributions do not change under permutations in $\mathbb{S}_\infty$.

(ii) **spreadable** if joint distributions do not change under such permutations in $\mathbb{S}_\infty$ which preserve the order of the variables considered.

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Of course $(i) \Rightarrow (ii)$. By a theorem of Ryll-Nardzewski (1957) we have equivalence if $\mathcal{A}$ is commutative.

Köstler's noncommutative de Finetti theorem

The noncommutative case gains interest from the following result.

Theorem (K)

Every spreadable sequence ι of random variables is stationary and conditionally independent (in the sense of commuting squares) over the tail algebra

$$\mathcal{A}^{\mathrm{tail}} := \bigcap_{n \geq 0} \mathrm{vN}(\iota_k(\mathcal{A}_0),\ k \geq n)$$

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We consider now the following question:

Are there (necessarily noncommutative) sequences of random variables which are spreadable but not exchangeable?

Artin's braid groups

Algebraic Definition (Artin 1925)

The **braid group** $\mathbb{B}_n$ is presented by $n-1$ generators $\sigma_1, \ldots, \sigma_{n-1}$ satisfying

$$\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \qquad \text{if } \mid i-j \mid = 1 \qquad \text{(B1)}$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad \text{if } \mid i-j \mid > 1 \qquad \text{(B2)}$$

$\mathbb{B}_1 \subset \mathbb{B}_2 \subset \mathbb{B}_3 \subset \ldots \subset \mathbb{B}_\infty$ (inductive limit)

Figure: Artin generators σ_i (left) and σ_i^{-1} (right)

Braidability

Definition (GK-B)

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$$\begin{aligned} \iota_n &= \rho(\sigma_n\sigma_{n-1}\cdots\sigma_1)\iota_0 && \text{if } n \geq 1 \\ \iota_0 &= \rho(\sigma_n)\iota_0 && \text{if } n \geq 2. \end{aligned}$$

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Observation:

There is a canonical homomorphism from $\mathbb{B}_\infty$ to $\mathbb{S}_\infty$ mapping the Artin generators of $\mathbb{B}_\infty$ to the Coxeter generators of $\mathbb{S}_\infty$.
If we replace $\mathbb{B}_\infty$ by $\mathbb{S}_\infty$ in the definition above, then what we get is equivalent to 'exchangeable'. Hence

$$\text{exchangeable} \Rightarrow \text{braidable}$$

An example: Gaussian representation

Choose $2 \leq p \in \mathbb{N}$ and a root of unity

$$\omega := \begin{cases} exp(2\pi\mathrm{i}/p) & \text{if p is odd;} \\ exp(\pi\mathrm{i}/p) & \text{if p is even.} \end{cases}$$

Then consider unitaries $(e_i)_{i\in\mathbb{N}_0}$ satisfying

$$\begin{aligned} e_i^p &= \mathbb{1} && \text{for all } i \\ e_i e_j &= \omega^2 e_j e_i && \text{whenever } i < j. \end{aligned}$$

Taking the weak closure of a matrix realization with respect to the trace we can think of $(e_i)_{i\in\mathbb{N}_0}$ as a sequence of random variables.

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exchangeable $\not\Leftarrow$ braidable

Constructive procedure

Suppose we are given a random variable

$$\iota_0 \colon (\mathcal{A}_0, \varphi_0) \to (\mathcal{A}, \varphi)$$

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If $\iota_0(\mathcal{A}_0)$ satisfies the **localization property**

$$\iota_0(\mathcal{A}_0) \subset \bigcap_{n\geq 2} \mathcal{A}^{\rho(\sigma_n)}$$

then we obtain a braidable random sequence by defining

$$\iota_n = \rho(\sigma_n \sigma_{n-1} \cdots \sigma_1)\iota_0 \quad \text{if } n \geq 1.$$

The maximal choice is $\iota_0(\mathcal{A}_0) = \bigcap_{n\geq 2} \mathcal{A}^{\rho(\sigma_n)}$.

Braidability implies Spreadability

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Theorem (K, GK-B)

(a) $\Rightarrow$ **(b)** $\Rightarrow$ **(c)** $\Rightarrow$ **(d)**

Braidability implies Spreadability (Proof by Example)

Note that a sequence ι of random variables is spreadable if and only if the (noncommutative) distribution is unchanged whenever the subscripts of the random variables change in an order preserving way.
For example we can change $1 < 2 < 4 > 1$ into $1 < 3 < 4 > 1$ and then we should have

$$\varphi(\iota_1(a)\,\iota_2(b)\,\iota_4(c)\,\iota_1(d)) = \varphi(\iota_1(a)\,\iota_3(b)\,\iota_4(c)\,\iota_1(d))$$

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Let us now assume that ι is braidable and prove the formula in this example from that.

Braidability implies Spreadability (Proof by Example)

Lemma (GK-B)

$$\rho(\sigma_k)\,\iota_m = \iota_m \quad \textit{if} \quad k \notin \{m, m+1\}$$

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Proof. Case $k > m+1$.

$$\begin{aligned}
\rho(\sigma_k)\iota_m &= \rho(\sigma_k)\rho(\sigma_m \ldots \sigma_1)\iota_0 \\
&= \rho(\sigma_m \ldots \sigma_1)\rho(\sigma_k)\iota_0 = \rho(\sigma_m \ldots \sigma_1)\iota_0 = \iota_m
\end{aligned}$$

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Case $k < m$.

$$\begin{aligned} \rho(\sigma_k)\iota_m &= \rho(\sigma_k)\rho(\sigma_m \dots \sigma_1)\iota_0 \\ &= \dots \rho(\sigma_k \sigma_{k+1} \sigma_k) \dots \iota_0 \\ &= \dots \rho(\sigma_{k+1} \sigma_k \sigma_{k+1}) \dots \iota_0 \\ &= \rho(\sigma_m \dots \sigma_1)\iota_0 = \iota_m \end{aligned}$$

Braidability implies Spreadability (Proof by Example)

Finally we can prove the formula in our example.

$$
\begin{aligned}
& \varphi(\iota_1(a)\ \iota_2(b)\ \iota_4(c)\ \iota_1(d)) \\
= \quad & \varphi(\rho(\sigma_3)\big(\iota_1(a)\,\iota_2(b)\,\iota_4(c)\,\iota_1(d)\big)) \\
= \quad & \varphi(\iota_1(a)\ \rho(\sigma_3)\big(\iota_2(b)\big)\ \iota_4(c)\,\iota_1(d)) \\
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To get concrete results from applications of the de Finetti theorem we need to identify the tail algebra. For braidable sequences the following result helps.

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Theorem (GK-B)

For a braidable sequence ι with $\iota_0(\mathcal{A}_0) = \mathcal{A}_0^\rho := \bigcap_{n \geq 2} \mathcal{A}^{\rho(\sigma_n)}$, we have

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In particular, these two algebras are trivial if the random sequence is conditionally independent over $\mathbb{C}$.

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In particular, these two algebras are trivial if the random sequence is conditionally independent over $\mathbb{C}$.

We can think of that as a braided version of the **Hewitt-Savage 0-1 law** known to probabilists which states a similar result about exchangeable sequences and the symmetric group $\mathbb{S}_\infty$.

Example: Left Regular Representation of $\mathbb{B}_\infty$

The **group von Neumann algebra** $L(\mathbb{B}_\infty)$ is generated by the left-regular representation $\{L_\sigma \mid \sigma \in \mathbb{B}_\infty\}$ of $\mathbb{B}_\infty$ on the Hilbert space $\ell^2(\mathbb{B}_\infty)$, where

$$L_\sigma f(\sigma') := f(\sigma^{-1}\sigma').$$

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Proposition (see GK-B)

- $L(\mathbb{B}_\infty)$ is a non-hyperfinite II_1-factor;
- $L(\mathbb{B}_\infty)$ does not have property T but it is not isomorphic to a free group factor.
- $L(\sigma_2, \sigma_3, \ldots) \subset L(\mathbb{B}_\infty)$ is an **irreducible** subfactor inclusion with infinite Jones index.

A new set of generators

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But we can apply our constructive procedure to σ_1 (using a 1-shifted adjoint left-regular representation $\sigma_n \mapsto \mathrm{Ad}\, L_{\sigma_{n+1}^{-1}}$ to fulfil the localization property) and obtain the following sequence of braids:

$$\gamma_1 := \sigma_1, \quad \gamma_2 := \sigma_1\sigma_2\sigma_1^{-1}, \ldots, \gamma_i := (\sigma_1\sigma_2\cdots\sigma_{i-1})\sigma_i(\sigma_{i-1}^{-1}\cdots\sigma_2^{-1}\sigma_1^{-1})$$

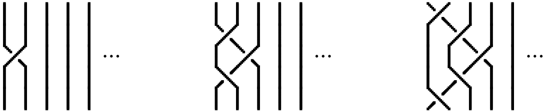

Figure: Braid diagrams of γ_1, γ_2 and γ_3 (left to right)

Square roots of free generators

The free group $\mathbb{F}_{n-1}$ is a subgroup of $\mathbb{B}_n$. Indeed, the following fact is well known to group theorists.

Proposition

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Proposition (GK-B)

The sequence (γ_i) is braidable (in $(L(\mathbb{B}_\infty), \mathrm{tr})$).
The tail algebra is equal to $\mathbb{C}$ and hence independence means factorization with respect to the trace.

Braided independence and freeness

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Theorem (GK-B)

Let $x \in \mathcal{B}_I$ and $y \in \mathcal{B}_J$ with $I \cap J = \emptyset$. Then

$$\mathrm{tr}(xy) = \mathrm{tr}(x)\,\mathrm{tr}(y).$$

> This factorization restricts to free group von Neumann algebras for $x \in \mathrm{vN}(\mathbb{F}_I) \subset \mathcal{B}_I$ and $y \in \mathrm{vN}(\mathbb{F}_J) \subset \mathcal{B}_J$.
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Question:
Is there a braided extension of the combinatorics of free probability?

Intermediate discussion

- One may continue by looking systematically at other braid group representations.
- To gain more experience we first looked at representations of the (infinite) symmetric group.
- Because

$$\text{exchangeable} \Rightarrow \text{braidable}$$

  all the general results obtained so far are still available!
- We found that we can give a new fully operator algebraic proof of a famous classical result by Thoma (1964).

Thoma's theorem

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Theorem (Thoma 64,)

An extremal character of the group $\mathbb{S}_\infty$ is of the form

$$\chi(\sigma) = \prod_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} a_i^k + (-1)^{k-1} \sum_{j=1}^{\infty} b_j^k \right)^{m_k(\sigma)} .$$

Here $m_k(\sigma)$ is the number of k-cycles in the permutation σ and the two sequences $(a_i)_{i=1}^{\infty}, (b_j)_{j=1}^{\infty}$ satisfy

$$a_1 \geq a_2 \geq \cdots \geq 0, \qquad b_1 \geq b_2 \geq \cdots \geq 0, \qquad \sum_{i=1}^{\infty} a_i + \sum_{j=1}^{\infty} b_j \leq 1.$$

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Characters and traces

Folklore result:
Every character χ of $\mathbb{S}_\infty$ gives rise to a unitary representation

$$\pi : \mathbb{S}_\infty \to \mathcal{U}(\mathcal{A}), \qquad \text{with } \mathcal{A} = \mathrm{vN}(\pi(\mathbb{S}_\infty))$$

such that there is a faithful tracial normal state tr on $\mathcal{A}$.
The converse is also true.
The character is extremal iff $\mathcal{A}$ is a factor.

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The transition to our setting is provided by considering the tracial probability space $(\mathcal{A}, \mathrm{tr})$ and the representation

$$\begin{array}{rcl} \rho\colon \mathbb{S}_\infty & \to & \mathrm{Aut}(\mathcal{A}, \mathrm{tr}) \\ \tau & \mapsto & \mathrm{Ad}(\pi(\tau)) \end{array}$$

Coxeter generators & star generators

Under the canonical homomorphism $\mathbb{B}_\infty \to \mathbb{S}_\infty$ the Artin generators go to the Coxeter generators $(i-1, i)$ of $\mathbb{S}_\infty$, where $\mathbb{S}_\infty$ acts on $\{0, 1, 2, 3, ...\}$ by permutations.

The (represented) Coxeter generators are not exchangeable.

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The (represented) Coxeter generators are not exchangeable.

We can find an exchangeable sequence by considering the images of the squares of free generators which we again denote by γ_i.

It turns out that $\gamma_i = (0, i)$.

Algebraists call them **star generators**.

Star generators & cycles

Lemma (Irving & Rattan '06, GK-S)

Let $k \geq 2$. A k-cycle $\sigma = (n_1, n_2, n_3, \ldots, n_k) \in \mathbb{S}_\infty$ is of the form

$$\sigma = \gamma_{n_1}\gamma_{n_2}\gamma_{n_3}\cdots\gamma_{n_{k-1}}\gamma_{n_k}\gamma_{n_1},$$

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where in the case $\sigma(0) \neq 0$ we put $n_1 = 0$ and define γ_0 to be the unit element.

Corollary (GK-S)

Disjoint cycles are supported by disjoint sets of star generators.

Cycles & Independence

- By $\mathcal{A}_n$ we denote the **fixed point algebra** in $\mathcal{A}$ of (the adjoint representation of) the subgroup $\mathbb{S}_{n+2,\infty}$ generated by

$$\sigma_i, \ i \geq n+2$$

In other words, $\mathcal{A}_n$ is the relative commutant of the
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- So we also need to identify the fixed point algebra $\mathcal{A}_0$ to make our independence results concrete.

Key observation

Lemma (GK-S)

Let $\gamma_{n_1}\gamma_{n_2}\gamma_{n_3}\cdots\gamma_{n_k}\gamma_{n_1}$ *be a* k*-cycle. Then*

$$E_0(\gamma_{n_1}\gamma_{n_2}\gamma_{n_3}\cdots\gamma_{n_k}\gamma_{n_1}) = \begin{cases} A_0^{k-1} & \text{if } n_1 = 0 \\ C_k & \text{if } n_1 \neq 0 \end{cases}$$

where

$$\begin{aligned} A_0 &:= E_0(\gamma_1) \\ C_k &:= E_{-1}(A_0^{k-1}). \end{aligned}$$

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We call these and similar objects **limit cycles** because they are weak limits (ergodic averages) of represented cycles. A systematic study is undertaken in [GK-S].

Sketch of proof

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The γ_i's are $\mathcal{A}_0$-independent. Thus $(L.H.S.) = E_0\Big(\gamma_{n_1}A_0^{k-1}\gamma_{n_1}\Big)$. But this equals $(R.H.S.)$, since for $x \in \mathcal{A}_0$ the $\gamma_i x \gamma_i$ are $\mathcal{A}_{-1}$-independent for different i. $\square$

Generators for fixed point algebras

Theorem (GK-S)

$$\mathcal{A}_0 = \mathrm{vN}(A_0, C_k \mid k \in \mathbb{N}) \qquad \text{(abelian!)}$$

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Further A_0 is trivial iff $\begin{cases} \text{the (subfactor) inclusion} \\ \mathrm{vN}(\mathbb{S}_{2,\infty}) \subset \mathrm{vN}(\mathbb{S}_\infty) \text{ is irreducible.} \end{cases}$

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Corollary (GK-S, Thoma Multiplicativity)

Let $m_k(\sigma)$ be the number of k-cycles in the cycle decomposition of the permutation $\sigma \in \mathbb{S}_\infty$. Then

$$E_{-1}(\sigma) = \prod_{k=2}^{\infty} C_k^{m_k(\sigma)}.$$

(Recall $C_k = E_{-1}(A_0^{k-1})$.)

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(Recall $C_k = E_{-1}(A_0^{k-1})$.)

If $\mathcal{A}$ is a factor, then E_{-1} can be replaced by the tracial state tr and we obtain

$$\operatorname{tr}(\sigma) = \prod_{k=2}^{\infty} \left[\operatorname{tr}(A_0^{k-1}) \right]^{m_k(\sigma)}$$

Spectral Theory

In particular, if $\mathcal{A}$ is a factor, then for a k-cycle σ

$$\mathrm{tr}(\sigma) = \mathrm{tr}(A_0^{k-1}) = \int t^{k-1}\, d\mu \ ,$$

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At this point there are some (not yet fully understood) similarities with Okounkov's approach and we derived some inspiration from his methods.

A. Okounkov, On representations of the infinite symmetric group. J. Math. Sci. 96(5), 3550-3589 (1999).

(electronic: `arXiv: math/9803037v1`)

Commuting squares & discrete spectrum

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Let $\mathcal{M}_0$ be a von Neumann subalgebra of the finite factor $\mathcal{M}$. Suppose the unitary $u \in \mathcal{M}$ satisfies:

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Corollary (Okounkov 99, GK-S)

Suppose $\mathrm{vN}(\mathbb{S}_\infty)$ is a factor. Then the limit 2-cycle $A_0 = E_0(\gamma_1)$ has discrete spectrum which may accumulate only at the point 0.

Sketch of Proof 1

Let μ be the spectral measure of $E_{\mathcal{M}_0}(u)$. For simplicity only, assume that it lives on $[0, 1]$.

For $\epsilon > 0$ and $B \subset [\epsilon, 1]$ a Borel set, let χ be the corresponding spectral projection of $E_{\mathcal{M}_0}(u)$.

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$$\epsilon\, \mu(B) \leq \mu(B)^{\frac{3}{2}}$$

From that

$$\mu(B) = 0 \quad \text{or} \quad \epsilon^2 \leq \mu(B)$$

Hence μ has no continuous part in $[\epsilon, 1]$.

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and now from the commuting square: $\quad = \mathrm{tr}(\chi)^3 = \mu(B)^3$

which gives the second inequality.

Thoma measures

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for every $k > 1$.

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That's it. Thank you.