FROM SUBFACTOR PLANAR ALGEBRAS TO SUBFACTORS

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Abstract. We present a purely planar algebraic proof of the main result of a paper of Guionnet-Jones-Shlaykhtenko which constructs an extremal subfactor from a subfactor planar algebra whose standard invariant is given by that planar algebra.

1. Introduction

This paper is being submitted for publication not for what it proves but how it proves it. What it does is to give a different proof of the main result of [GnnJnsShl] which, in turn, offers an alternative proof of an important result of Popa in [Ppa] that may be paraphrased as saying that any subfactor planar algebra arises from an extremal subfactor.

In this introductory section, we will briefly review the main result of [GnnJnsShl] and the ingredients of its proof and compare and contrast the proof presented here with that one.

The main result of [GnnJnsShl] begins with the construction of a tower $Gr_k(P)$ of graded $*$-algebras with compatible traces Tr_k associated to a subfactor planar algebra P. Next, appealing to a result in [PpaShl], the planar algebra P is considered as embedded as a planar subalgebra of the planar algebra of a bipartite graph as described in [Jns2]. It is shown that that the traces Tr_k are faithful and positive and then II_1-factors M_k are obtained by appropriate completions (in case the modulus $\delta > 1$ - which is the only real case of interest). It is finally seen that the tower of M_k's is the basic construction tower for (the finite index, extremal II_1-subfactor) $M_0 \subseteq M_1$ and that the planar algebra of this subfactor is naturally isomorphic to the original planar algebra P. The proofs all rely on techniques of free probability and random matrices and indeed, one of the stated goals of the paper is to demonstrate the connections between these and planar algebras.

The *raison d'être* of this paper is to demonstrate the power of planar algebra techniques. We begin with a short summary of our notations and conventions regarding planar algebras in Section 2. In Section 3, we describe a tower $F_k(P)$ of filtered $*$-algebras, with compatible traces and 'conditional expectations', associated to a subfactor planar algebra P. The positivity of the traces being obvious, we show in Section 4 that the associated GNS representations are bounded and thus yield a tower of finite von Neumann algebra completions. The heart of this paper is Section 5 which is devoted to showing that these completions are factors and to computations of some relative commutants. The penultimate Section 6 identifies the tower as a basic construction tower of a finite index extremal subfactor with associated planar algebra as the original P. The final section exhibits interesting

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trace preserving $*$-isomorphisms between the algebras $Gr_k(P)$ of [GnnJnsShl] and our $F_k(P)$, thus justifying - to some extent - the first sentence of this paper.

All our proofs rely solely on planar algebra techniques and in that sense our paper is mostly self-contained. We will need neither the embedding theorem for a subfactor planar algebra into the planar algebra of a bipartite graph nor any free probability or random matrix considerations. In the trade-off between analytic techniques and algebraic/combinatorial techniques that is characteristic of subfactor theory, it would be fair to say that [GnnJnsShl] leans towards the analytic approach while this paper takes the opposite tack.

After this paper had been written, we communicated it to Jones requesting his comments. He wrote back saying that this "may be similar to [JnsShlWlk]" which appears on his homepage; and we discovered that this is indeed the case.

2. Subfactor planar algebras

The purpose of this section is to fix our notations and conventions regarding planar algebras. We assume that the reader is familiar with planar algebras as in [Jns] or in [KdySnd] so we will be very brief.

Recall that the basic structure that underlies planar algebras is an action by the 'coloured operad of planar tangles' which concept we will now explain. Consider the set $Col = \{0_+, 0_-, 1, 2, \cdots\}$, whose elements are called colours. We will not define a tangle but merely note the following features. Each tangle has an external box, denoted D_0, and a (possibly empty) ordered collection of internal non-nested boxes denoted D_1, D_2, $\cdots$. Each box has an even number (again possibly 0) of points marked on its boundary. A box with $2n$ points on its boundary is called an n-box or said to be of colour n.

There is also given a collection of disjoint curves each of which is either closed, or joins a marked point on one of the boxes to another such. For each box having at least one marked point on its boundary, one of the regions (= connected components of the complement of the boxes and curves) that impinge on its boundary is distinguished and marked with a $*$ placed near its boundary. The whole picture is to be planar and each marked point on a box must be the end-point of one of the curves. Finally, there is given a chequerboard shading of the regions such that the $*$-region of any box is shaded white. A 0-box is said to be 0_+ box if the region touching its boundary is white and a 0_- box otherwise. A 0 without the $\pm$ qualification will always refer to 0_+. A tangle is said to be an n-tangle if its external box is of colour n. Tangles are defined only upto a planar isotopy preserving the $*$'s, the shading and the ordering of the internal boxes.

We illustrate several important tangles in Figure 1. This figure (along with others in this paper) uses the following notational device for convenience in drawing tangles. A strand in a tangle with a *non-negative integer*, say t, adjacent to it will indicate a t-cable of that strand, i.e., a parallel cable of t strands, in place of the one actually drawn. Thus for instance, the tangle equations of Figure 2 hold. In the sequel, we will have various integers adjacent to strands in tangles and leave it to the reader to verify in each case that the labelling integers are indeed non-negative.

A useful labelling convention for tangles (that we will not be very consistent in using though) is to decorate its tangle symbol, such as I, EL, M or TR, with subscripts and a superscript that give the colours of its internal boxes and external box respectively. With this, we may dispense with showing the shading, which

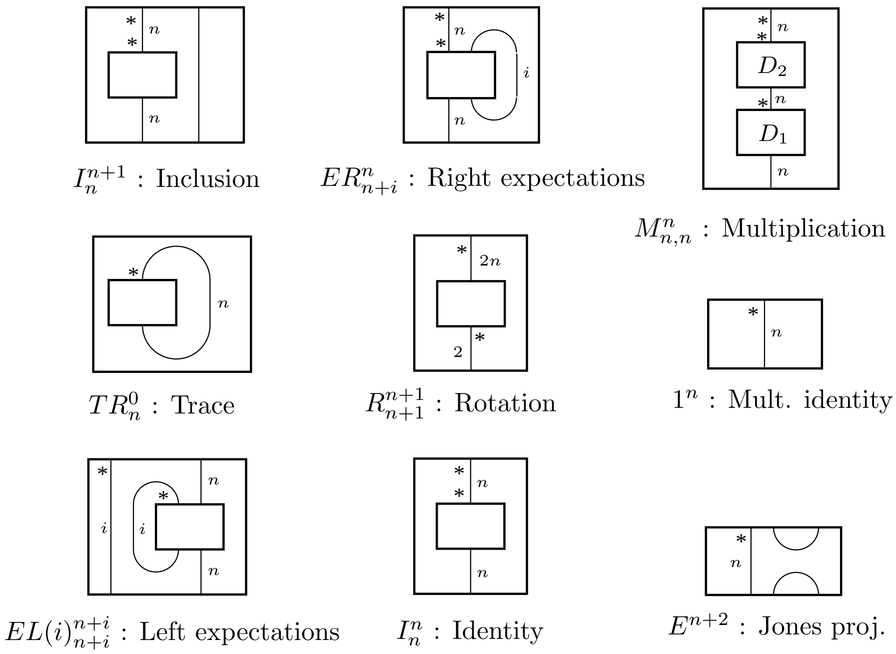


FIGURE 1. Some important tangles (defined for $n \geq 0, 0 \leq i \leq n$).

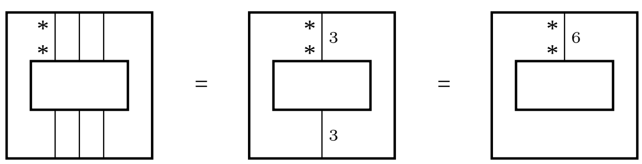


FIGURE 2. Illustration of cabling notation for tangles

is then unambiguously determined. Another useful device is to label the marked points on any n-box with the numbers $1, 2, \cdots, 2n-1, 2n$ in a clockwise fashion so that the interval from $2n$ to 1 is in its $*$-region.

The basic operation that one can perform on tangles is substitution of one into a box of another. If T is a tangle that has some internal boxes $D_{i_1}, \cdots, D_{i_j}$ of colours $n_{i_1}, \cdots, n_{i_j}$ and if $S_1, \cdots, S_j$ are arbitrary tangles of colours $n_{i_1}, \cdots, n_{i_j}$, then we may substitute S_t into the box D_{i_t} of T for each t - such that the '$*$'s match' - to get a new tangle that will be denoted $T \circ_{(D_{i_1}, \cdots, D_{i_j})} (S_1, \cdots, S_j)$. The collection of tangles along with the substitution operation is called the coloured operad of planar tangles.

A planar algebra P is an algebra over the coloured operad of planar tangles. By this, is meant the following: P is a collection $\{P_n\}_{n \in Col}$ of vector spaces and linear maps $Z_T : P_{n_1} \otimes P_{n_2} \otimes \cdots \otimes P_{n_b} \to P_{n_0}$ for each n_0-tangle T with internal boxes of colours $n_1, n_2, \cdots, n_b$. The collection of maps is to be 'compatible with substitution of tangles and renumbering of internal boxes' in an obvious manner. For a planar algebra P, each P_n acquires the structure of an associative, unital algebra with multiplication defined using the tangle $M_{n,n}^n$ and unit defined to be $1_n = Z_{1^n}(1)$.

Among planar algebras, the ones that we will be interested in are the subfactor planar algebras. These are complex, finite-dimensional and connected in the sense that each P_n is a finite-dimensional complex vector space and $P_{0_\pm}$ are one dimensional. They have a positive modulus δ, meaning that closed loops in a tangle T contribute a multiplicative factor of δ in Z_T. They are spherical in that for a 0-tangle T, the function Z_T is not just planar isotopy invariant but also an isotopy invariant of the tangle regarded as embedded on the surface of the two sphere. Further, each P_n is a C^*-algebra in such a way that for an n_0-tangle T with internal boxes of colours $n_1, n_2, \cdots, n_b$ and for $x_i \in P_{n_i}$, the equality $Z_T(x_1 \otimes \cdots \otimes x_b)^* \ = \ Z_{T^*}(x_1^* \otimes \cdots \otimes x_b^*)$ holds, where T^* is the adjoint of the tangle T - which, by definition, is obtained from T by reflecting it. Finally, the trace $\tau : P_n \rightarrow \mathbb{C} = P_0$ defined by:

$$\tau(x) \ = \ \delta^{-n} Z_{TR_n^0}(x)$$

is postulated to be a faithful, positive (normalised) trace for each $n \geq 0$.

We then have the following fundamental thorem of Jones in [Jns].

Theorem 2.1. *Let*

$$(M_0 =)N \subset M (= M_1) \subset^{e_2} M_2 \subset \cdots \subset^{e_n} M_n \subset^{e_{n+1}} \cdots$$

be the tower of the basic construction associated to an extremal subfactor with index $[M : N] = \delta^2 < \infty$. Then there exists a unique subfactor planar algebra $P = P^{N \subset M}$ of modulus δ satisfying the following conditions:

*(0) $P_n^{N \subset M} = N' \cap M_n \ \forall n \geq 0$ - where this is regarded as an equality of *-algebras which is consistent with the inclusions on the two sides;*

(1) $Z_{E^{n+1}}(1) = \delta \ e_{n+1} \ \forall \ n \geq 1$;

(2) $Z_{EL(1)_{n+1}^{n+1}}(x) = \delta \ E_{M' \cap M_{n+1}}(x) \ \forall \ x \in N' \cap M_{n+1}, \ \forall n \geq 0$;

(3) $Z_{ER_{n+1}^n}(x) = \delta \ E_{N' \cap M_n}(x) \ \forall \ x \in N' \cap M_{n+1}$; and this (suitably interpreted for $n = 0_\pm$) is required to hold for all $n \in Col$.

Conversely, any subfactor planar algebra P with modulus δ arises from an extremal subfactor of index δ^2 in this fashion. □

Recall that a finite index II_1-subfactor $N \subseteq M$ is said to be extremal if the restriction of the traces on N' (computed in $\mathcal{L}(L^2(M))$) and M to $N' \cap M$ agree. The notations $E_{M' \cap M_{n+1}}$ and $E_{N' \cap M_n}$ stand for the trace preserving conditional expectations of $N' \cap M_{n+1}$ onto $M' \cap M_{n+1}$ and $N' \cap M_n$ respectively.

The converse part of Jones' theorem is, in essence, the result of Popa alluded to in the introduction, as remarked in [Jns]. It is this converse, as proved in [GnnJnsShl], for which we supply a different proof in the rest of this paper.

We note that the multiplication tangle here agrees with the one in [GnnJnsShl] but is adjoint to the ones in [Jns] and in [KdySnd] while the rotation tangle is as in [KdySnd] but adjoint to the one in [Jns].

Remark 2.2. For a subfactor planar algebra P, (i) the right expectation tangles ER_{n+i}^n define surjective positive maps $P_{n+i} \rightarrow P_n$ of norm δ^i, (ii) the left expectation tangles $EL(i)_{n+1}^{n+1}$ define positive maps $P_{n+1} \rightarrow P_{n+1}$ whose images, denoted $P_{i,n+1}$, are C^*-subalgebras of P_{n+1} and (iii) the rotation tangles R_{n+1}^{n+1} define unitary maps $P_{n+1} \rightarrow P_{n+1}$. The first two statements follow from the fact that ER_{n+i}^n

and $EL(i)_{n+1}^{n+1}$ give (appropriately scaled) conditional expectation maps that preserve a faithful, positive trace while the third is a consequence of the compatibility of the tangle $*$ and the $*$ of the C^*-algebra P_{n+1}.

3. The tower of filtered $*$-algebras with trace

For the rest of this paper, the following notation will hold. Let P be a subfactor planar algebra of modulus $\delta > 1$. We therefore have finite-dimensional C^*-algebras P_n for $n \in Col$ with appropriate inclusions. For $x \in P_n$, set $||x||_{P_n} = \tau(x^*x)^{\frac{1}{2}}$; this defines a norm on P_n.

For $k \geq 0$, let $F_k(P)$ be the vector space direct sum $\oplus_{n=k}^{\infty} P_n$ (where $0 = 0_+$, here and in the sequel). Our goal, in this section, is to equip each $F_k(P)$ with a filtered, associative, unital $*$-algebra structure with normalised trace t_k and to describe trace preserving filtered $*$-algebra inclusions $F_0(P) \subseteq F_1(P) \subseteq F_2(P) \subseteq \cdots$, as well as conditional expectation-like maps $F_0(P) \stackrel{E_0}{\leftarrow} F_1(P) \stackrel{E_1}{\leftarrow} F_2(P) \stackrel{E_2}{\leftarrow} \cdots$.

We begin by defining the multiplication. For $a \in F_k(P)$, we will denote by a_n, its P_n component for $n \geq k$. Thus $a = \sum_{n=k}^{\infty} a_n = (a_k, a_{k+1}, \cdots) \in F_k(P)$, where only finitely many a_n are non-zero. Now suppose that $a = a_m \in P_m$ and $b = b_n \in P_n$ where $m, n \geq k$. Their product in $F_k(P)$, denoted[1] $a\#b$, is defined to be $\sum_{t=|n-m|+k}^{n+m-k} (a\#b)_t$ where $(a\#b)_t$ is given by the tangle in Figure 3. Define $\#$

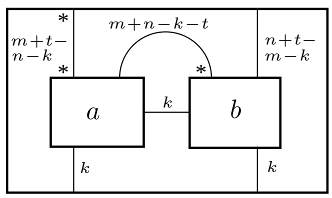


Figure 3. Definition of the P_t component of $a\#b$.

by extending this map bilinearly to the whole of $F_k(P) \times F_k(P)$.

As in [GnnJnsShl], we will reserve '$*$' to denote the usual involution on P_n's and use $\dagger$ (rather than $\dagger_k$) to denote the involution on $F_k(P)$. For $a = a_m \in P_m \subseteq F_k(P)$ define $a^\dagger \in P_m$ by the tangle in Figure 4 and extend additively to the whole

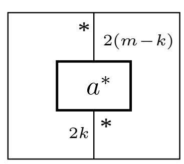


Figure 4. Definition of the $*$-structure on $F_k(P)$.

of $F_k(P)$. Note that $a^\dagger = Z_{(R_m^m)^k}(a^*)$ where R_m^m is the m-rotation tangle and $(R_m^m)^k = R_m^m \circ R_m^m \circ \cdots \circ R_m^m$ (k factors).

Next, define the linear functional t_k on $F_k(P)$ to be the normalised trace of its P_k component, i.e., for $a = (a_k, a_{k+1}, \cdots) \in F_k(P)$ define $t_k(a) = \tau(a_k)$.

[1]Rather than being notationally correct and write $\#_k$, we drop the subscript in the interests of aesthetics.

Then, define the 'inclusion map' of $F_{k-1}(P)$ into $F_k(P)$ (for $k \geq 1$) as the map whose restriction takes $P_{n-1} \subseteq F_{k-1}(P)$ to $P_n \subseteq F_k(P)$ by the tangle illustrated in Figure 5.

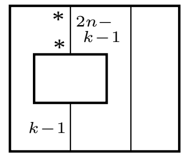


FIGURE 5. The inclusion from $P_{n-1} \subseteq F_{k-1}(P)$ to $P_n \subseteq F_k(P)$.

Finally, define the conditional expectation-like map $E_{k-1} : F_k(P) \to F_{k-1}(P)$ (for $k \geq 1$) as δ^{-1} times the map whose restriction takes $P_n \subseteq F_k(P)$ to $P_{n-1} \subseteq F_{k-1}(P)$ by the tangle illustrated in Figure 6.

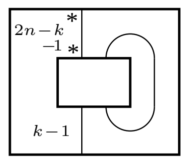


FIGURE 6. Definition of δE_{k-1} from $P_n \subseteq F_k(P)$ to $P_{n-1} \subseteq F_{k-1}(P)$.

An important observation that we use later is that 'restricted' to $P_k \subseteq F_k(P)$, all these maps are the 'usual' ones of the planar algebra P. More precisely we have (a) $\#|_{P_k \otimes P_k} = Z_{M^k_{k,k}}$, (b) $\dagger|_{P_k} = *$, (c) $t_k|_{P_k} = \tau$, (d) The inclusion of $F_{k-1}(P)$ into $F_k(P)$ restricts to $Z_{I^k_{k-1}} : P_{k-1} \to P_k$, and (e) $\delta E_{k-1}|_{P_k} = Z_{ER^{k-1}_k}$.

We can now state the main result of this section.

Proposition 3.1. *Let $k \geq 0$. Then, the following statements hold.*

1. *The vector space $F_k(P)$ acquires a natural associative, unital, filtered algebra structure for the # multiplication.*
2. *The operation † defines a conjugate linear, involutive, anti-isomorphism of $F_k(P)$ to itself thus making it a $*$-algebra.*
3. *The map t_k on $F_k(P)$ satisfies the equation $t_k(b^\dagger \# a) = \delta^{-k} \sum_{n=k}^{\infty} \delta^n \tau(b_n^* a_n)$ and consequently defines a normalised trace on $F_k(P)$ that makes $\langle a|b\rangle = t_k(b^\dagger \# a)$ an inner-product on $F_k(P)$.*
4. *The inclusion map of $F_k(P)$ into $F_{k+1}(P)$ is a normalised trace-preserving unital $*$-algebra monomorphism.*
5. *The map $E_k : F_{k+1}(P) \to F_k(P)$ is a $*$- and trace-preserving $F_k(P)-F_k(P)$ bimodule retraction for the inclusion map of $F_k(P)$ into $F_{k+1}(P)$.*

Proof. (1) Take $a \in P_m, b \in P_n, c \in P_p$ with $m, n, p \geq k$. Then, by definition of #,

$$(a\#b)\#c = \sum_{t=|m-n|+k}^{m+n-k} (a\#b)_t \# c = \sum_{t=|m-n|+k}^{m+n-k} \sum_{s=|t-p|+k}^{t+p-k} ((a\#b)_t \# c)_s, \text{ while}$$

$$a\#(b\#c) = \sum_{v=|n-p|+k}^{n+p-k} a\#(b\#c)_v = \sum_{v=|n-p|+k}^{n+p-k} \sum_{u=|m-v|+k}^{m+v-k} (a\#(b\#c)_v)_u.$$

Let $I(m,n,p) = \{(t,s) : |m-n|+k \le t \le m+n-k, |t-p|+k \le s \le t+p-k\}$ and $J(m,n,p) = \{(v,u) : |n-p|+k \le v \le n+p-k, |m-v|+k \le u \le m+v-k\}$, so that the former indexes terms of $(a\#b)\#c$ while the latter indexes terms of $a\#(b\#c)$.

A routine verification shows that (a) $I(m,n,p) = J(p,n,m)$ and (b) the map $T(m,n,p) : I(m,n,p) \to J(m,n,p)$ defined by $T(m,n,p)((t,s)) = (max\{m+p, n+s\} - t, s)$ is a well-defined bijection with inverse $T(p,n,m)$ such that (c) if $T(m,n,p)((t,s)) = (v,u)$, then both $((a\#b)_t\#c)_s$ and $(a\#(b\#c)_v)_u$ are equal to the figure on the right or on the left in Figure 7 according as $m+p \ge n+s$ or

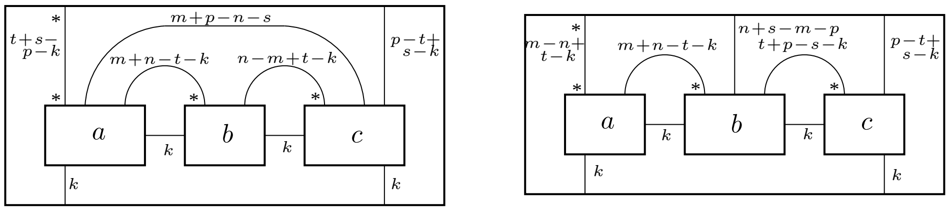


FIGURE 7. $((a\#b)_t\#c)_s = (a\#(b\#c)_v)_u$.

$m+p \le n+s$; this finishes the proof of associativity.

Observe that the usual unit 1_k of P_k is also the unit for the #-multiplication in $F_k(P)$ and that there is an obvious filtration of $F_k(P)$ by the subspaces which are the direct sums of the first n components of $F_k(P)$ for $n \ge 0$.

(2) This is an entirely standard pictorial planar algebra argument which we'll omit.

(3) By definition, $t_k(b^\dagger\#a)$ is the normalised trace of $(b^\dagger\#a)_k$. By definition of #, the k-component of $b^\dagger\#a$ has contributions only from $b_n^\dagger\#a_n$ for $n \ge k$, and the normalised trace of this contribution is given by $\delta^{-k+n}\tau(b_n^*a_n)$, as needed. The fact that t_k is a normalised trace on $F_k(P)$ follows from this and the unitarity of $Z_{R_n^n}$ on P_n. It also follows that $\langle a|b\rangle = t_k(b^\dagger\#a)$ is an inner-product on $F_k(P)$.

(4), (5) We also omit these proofs which are routine applications of pictorial techniques using the definitions. $\square$

We define H_k to be the Hilbert space completion of $F_k(P)$. Note that for the inner-product on $F_k(P)$, the subspaces P_n of H_k are mutually orthogonal and so H_k is their orthogonal direct sum $\oplus_{n=k}^\infty P_n$. In particular, elements of H_k are sequences $\xi = (x_k, x_{k+1}, \cdots)$ where $||\xi||_{H_k}^2 = \delta^{-k}\sum_{n=k}^\infty \delta^n\tau(x_n^*x_n) < \infty$.

4. Boundedness of the GNS representations

Our goal in this section is to show that for each $k \ge 0$, the left and right regular representations of $F_k(P)$ are both bounded for the norm on $F_k(P)$ and therefore extend uniquely to $*$-homomorphisms $\lambda_k, \rho_k : F_k(P) \to \mathcal{L}(H_k)$. We then show that the von Neumann algebras generated by $\lambda_k(F_k(P))$ and $\rho_k(F_k(P))$ are finite and commutants of each other. Finally we show that for $k \ge 0$, the finite von Neumann algebras $\lambda_k(F_k(P))''$ naturally form a tower.

The key estimate we need for proving boundedness is contained in Proposition 4.2, the proof of which appeals to the following lemma.

Lemma 4.1. *Fix $p \ge k$. Suppose that $0 \le q \le 2p$, $0 \le i \le 2p-q$ and $a \in P_p \subseteq H_k$. There is a unique positive $c \in P_{2p-q}$ such that the equation of Figure 8 holds. Further, $||c||_{H_k}^2 = \delta^i||a||_{H_k}^2$.*

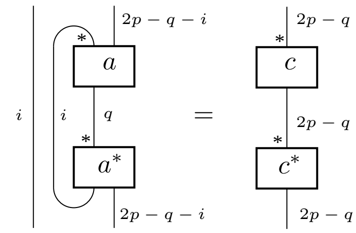


FIGURE 8. On positivity

Proof. To show the existence and uniqueness of c, it clearly suffices to see that the picture on the left in Figure 8 defines a positive element of the C^*-algebra P_{2p-q}, for then, c must be its positive square root. But this element may be written as $Z_{EL(i)^{2p-q}_{2p-q}}(b)$ where b is illustrated in Figure 9, and so by Remark 2.2 it suffices to see that b itself is positive. The pictures on the right in Figure 9 exhibit b as (i)

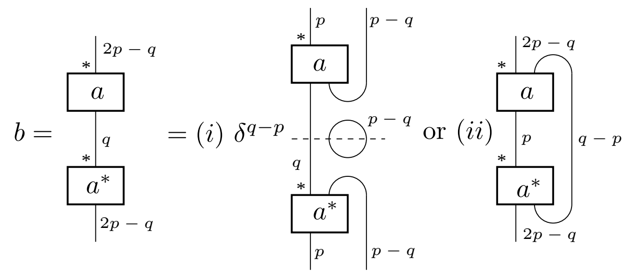


FIGURE 9. Positivity of b

a positive multiple of a product of an element of P_{2p-q} (the one below the dotted line) and its adjoint (the one above) if $q \leq p$ or as (ii) $Z_{ER_p^{2p-q}}(a^*a)$ (the image of a positive element under a positive map) if $q \geq p$, proving that b is positive in either case. The norm statement follows by applying the trace tangle TR^0_{2p-q} on both sides in Figure 8 and recalling that $\delta^{k-u}||x||^2_{H_k} = \tau(x^*x)$ for $x \in P_u \subseteq H_k$. □

Proposition 4.2. *Suppose that $a = a_m \in P_m \subseteq F_k(P)$. There exists a constant K (depending only on a) so that, for all $b = b_n \in P_n \subseteq F_k(P)$ and all t such that $|m-n|+k \leq t \leq m+n-k$, we have $||(a\#b)_t||_{H_k} \leq K||b||_{H_k}$.*

Proof. By definition, $||(a\#b)_t||^2_{H_k} = \delta^{-k}\delta^t\tau((a\#b)_t(a\#b)^*_t)$, and so $\delta^k||(a\#b)_t||^2_{H_k}$ is given by - using Figure 3 - the value of the tangle on the left in Figure 10. Setting $\tilde{b} = Z_{(R^n_n)^{-k}}(b)$, this is also seen to be equal to the value of the tangle in the middle in Figure 10.

Now let $u = m+n+k-t$ and note that $2k \leq u \leq min\{2m, 2n\}$. We now apply Lemma 4.1 to the tangle on the left of the dotted line (with $i = 0$, $p = m$ and $q = m-n+t-k$) and to the (inverted) tangle on the right of the dotted line (with $i = k$, $p = n$ and $q = n+t-m-k$) to conclude that there exist $c_u, d_u \in P_u$ such that $||c_u||^2_{H_k} = ||a||^2_{H_k}$, $||d_u||^2_{H_k} = \delta^k||\tilde{b}||^2_{H_k} = \delta^k||b||^2_{H_k}$ and the second equality in Figure 10 holds. Therefore, $||(a\#b)_t||^2_{H_k} = \delta^{-k}\delta^u\tau(c^*_uc_ud_ud^*_u) = \delta^{u-k}||c_ud_u||^2_{P_u}$

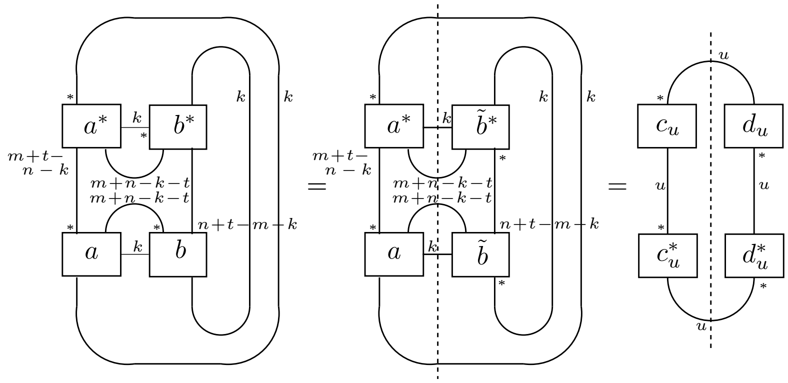


FIGURE 10. $\delta^k||(a\#b)_t||^2_{H_k}$

$\leq \delta^{u-k}||c_u||^2_{op}||d_u||^2_{P_u} = ||c_u||^2_{op}||d_u||^2_{H_k} = \delta^k||c_u||^2_{op}||b||^2_{H_k}$. Here, we write $||x||^2_{P_u} = \tau(x^*x)$ and $||x||_{op}$ for the operator norm of x in the C^*-algebra P_u.

Finally, take K to be the maximum of $\delta^{\frac{k}{2}}||c_u||_{op}$ as u varies between $2k$ and $2m$, which clearly depends only on a. □

We are ready to prove boundedness of the left regular representation. During the course of the proof we will need to use the fact that $||\sum_{i=1}^k a_i||^2 \leq k(\sum_{i=1}^k ||a_i||^2)$ for vectors $a_1, \cdots, a_k$ in an inner-product space, which follows on applying Cauchy-Schwarz to the vectors $(||a_1||, \cdots, ||a_k||)$ and $(1, 1, \cdots, 1)$ in $\mathbb{C}^k$.

Proposition 4.3. *Suppose that $a \in P_m \subseteq F_k(P)$ and $\lambda_k(a) : F_k(P) \to F_k(P)$ is defined by $\lambda_k(a)(b) = a\#b$. Then there is a constant C (depending only on a) so that $||\lambda_k(a)(b)||_{H_k} \leq C||b||_{H_k}$ for all $b \in F_k(P)$.*

Proof. Suppose that $b = \sum_{n=k}^\infty b_n$ (the sum being finite, of course). Then $a\#b = \sum_{t=k}^\infty \sum_{n=k}^\infty (a\#b_n)_t = \sum_{t=k}^\infty \sum_{n=|t-m|+k}^{t+m-k} (a\#b_n)_t$, where the last equality is by definition of where non-zero terms of $a\#b_n$ may lie. Thus $(a\#b)_t = \sum_{n=|t-m|+k}^{t+m-k} (a\#b_n)_t$ is a sum of at most $1 + min\{2(t-k), 2(m-k)\}$ terms. By the remark preceding the proposition, it follows that

$$
\begin{aligned}
||(a\#b)_t||^2_{H_k} &\leq (1 + 2(m-k)) \sum_{n=|t-m|+k}^{t+m-k} ||(a\#b_n)_t||^2_{H_k} \\
&\leq K^2(1 + 2(m-k)) \sum_{n=|t-m|+k}^{t+m-k} ||b_n||^2_{H_k}
\end{aligned}
$$

where K is as in Proposition 4.2. Therefore,

$$
\begin{aligned}
||a\#b||^2_{H_k} &= \sum_{t=k}^{\infty} ||(a\#b)_t||^2_{H_k} \\
&\leq K^2(1+2(m-k)) \sum_{t=k}^{\infty} \sum_{n=|t-m|+k}^{t+m-k} ||b_n||^2_{H_k} \\
&= K^2(1+2(m-k)) \sum_{n=k}^{\infty} \sum_{t=|n-m|+k}^{n+m-k} ||b_n||^2_{H_k} \\
&= K^2(1+2(m-k)) \sum_{n=k}^{\infty} (1+min\{2(n-k),2(m-k)\})||b_n||^2_{H_k} \\
&\leq K^2(1+2(m-k))^2||b||^2_{H_k}.
\end{aligned}
$$

So we may choose C to be $K(1+2(m-k))$. □

It follows from Proposition 4.3 that for any $a \in F_k(P)$, the map $\lambda_k(a)$ is bounded and thus extends uniquely to an element of $\mathcal{L}(H_k)$. We then get a $*$-representation $\lambda_k : F_k(P) \to \mathcal{L}(H_k)$.

In order to avoid repeating hypotheses we make the following definition. By a finite pre-von Neumann algebra, we will mean a complex $*$-algebra A that is equipped with a normalised trace t such that (i) the sesquilinear form defined by $\langle a'|a\rangle = t(a^*a')$ defines an inner-product on A and such that (ii) for each $a \in A$, the left-multiplication map $\lambda_A(a) : A \to A$ is bounded for the trace induced norm of A. Examples that we will be interested in are the $F_k(P)$ with their natural traces t_k for $k \geq 0$. We then have the following simple lemma which motivates the terminology and whose proof we sketch in some detail.

Lemma 4.4. *Let A be a finite pre-von Neumann algebra with trace t_A, and H_A be the Hilbert space completion of A for the associated norm, so that the left regular representation $\lambda_A : A \to \mathcal{L}(H_A)$ is well-defined, i.e., for each $a \in A$, $\lambda_A(a) : A \to A$ extends to a bounded operator on H_A. Then,*

(0) *The right regular representation $\rho_A : A \to \mathcal{L}(H_A)$ is also well-defined.*

Let $M_A^\lambda = \lambda_A(A)''$ and $M_A^\rho = \rho_A(A)''$. Then the following statements hold:

1. *The 'vacuum vector' $\Omega_A \in H_A$ (corresponding to $1 \in A \subseteq H_A$) is cyclic and separating for both M_A^λ and M_A^ρ.*
2. *M_A^λ and M_A^ρ are commutants of each other.*
3. *The trace t_A extends to faithful, normal, tracial states t_A^λ and t_A^ρ on M_A^λ and M_A^ρ respectively.*
4. *H_A can be identified with the standard modules $L^2(M_A^\lambda, t_A^\lambda)$ and $L^2(M_A^\rho, t_A^\rho)$.*

Proof. Start with the modular conjugation operator J_A which is the unique bounded extension, to H_A, of the involutive, conjugate-linear, isometry defined on the dense subspace $A\Omega_A \subseteq H_A$ by $a\Omega_A \mapsto a^*\Omega_A$, and which satisfies $J_A = J_A^* = J_A^{-1}$ (where J^* is defined by $\langle J^*\xi|\eta\rangle = \langle J\eta|\xi\rangle$ for conjugate linear J).
(0) Note that $J_A\lambda_A(a^*)J_A(\tilde{a}\Omega_A) = J_A\lambda_A(a^*)(\tilde{a}^*\Omega_A) = J_A(a^*\tilde{a}^*\Omega_A) = \tilde{a}a\Omega_A = \rho_a(\tilde{a}\Omega_A)$ and so for each $a \in A$, $\rho_A(a) : A \to A$ extends to the bounded operator $J_A\lambda_A(a^*)J_A$ on H. The map $\lambda_A : A \to \mathcal{L}(H)$ is a $*$-homomorphism while the map $\rho_A : A \to \mathcal{L}(H)$ is a $*$-anti-homomorphism.

(1) Since $\lambda_A(A) \subseteq \mathcal{L}(H_A)$ and $\rho_A(A) \subseteq \mathcal{L}(H_A)$ clearly commute, so do M_A^λ and M_A^ρ. So each is contained in the commutant of the other and it follows easily from definitions that Ω_A is cyclic and hence separating for both M_A^λ and M_A^ρ.
(2) Observe now that the subspace $K = \{x \in \mathcal{L}(H_A) : J_A x\Omega_A = x^*\Omega_A\}$ is weakly closed and contains M_A^λ, M_A^ρ and their commutants. (Reason: That K is weakly closed and contains $\lambda_A(A)$ and $\rho_A(A)$ is obvious, and so K also contains their weak closures M_A^λ and M_A^ρ. But now, for $x' \in (M_A^\lambda)' \cup (M_A^\rho)'$ and $a\Omega_A \in A\Omega_A$, we have

$$\begin{aligned} \langle J_A x'\Omega_A | a\Omega_A\rangle &= \langle J_A a\Omega_A | x'\Omega_A\rangle \\ &= \langle a^*\Omega_A | x'\Omega_A\rangle \\ &= \langle \Omega_A | x' a\Omega_A\rangle \\ &= \langle x'^*\Omega_A | a\Omega_A\rangle, \end{aligned}$$

where the third equality follows from interpreting $a^*\Omega_A$ as either $\lambda_A(a)^*\Omega_A$ or $\rho_A(a)^*\Omega_A$ according as $x' \in (M_A^\lambda)'$ or $x' \in (M_A^\rho)'$. Density of $A\Omega_A$ in H now implies that $J_A x'\Omega_A = x'^*\Omega_A$).

Since $K \supseteq M_A^\lambda$, it is easy to see that $J_A M_A^\lambda J_A \subseteq (M_A^\lambda)'$, and similarly since $K \supseteq (M_A^\lambda)'$, we have $J_A (M_A^\lambda)' J_A \subseteq (M_A^\lambda)'' = M_A^\lambda$. Comparing, we find indeed that $J_A M_A^\lambda J_A = (M_A^\lambda)'$. On the other hand, taking double commutants of $J_A \lambda_A(A) J_A = \rho_A(A)$ gives $J_A M_A^\lambda J_A = M_A^\rho$ and so $(M_A^\lambda)' = M_A^\rho$.
(3) Define a linear functional t on $\mathcal{L}(H_A)$ by $t(x) = \langle x\Omega_A | \Omega_A\rangle$ and observe that this extends the trace t_A on A where A is regarded as contained in $\mathcal{L}(H_A)$ via either λ_A or ρ_A. Define $t_A^\lambda = t|_{M_A^\lambda}$ and $t_A^\rho = t|_{M_A^\rho}$. A little thought shows that these are traces on M_A^λ and M_A^ρ. Positivity is clear and faithfulness is a consequence of the fact that Ω_A is separating for M_A^λ and M_A^ρ. Normality (= continuity for the σ-weak topology) holds since t is a vector state on $\mathcal{L}(H_A)$.
(4) This is an easy consequence of (3). $\square$

We conclude that if $M_k^\lambda = \lambda_k(F_k(P))'' \subseteq \mathcal{L}(H_k)$ and $M_k^\rho = \rho_k(F_k(P))'' \subseteq \mathcal{L}(H_k)$ and $\Omega_k \in H_k$ is the vacuum vector $1 = 1_k \in P_k \subseteq F_k(P) \subseteq H_k$, then M_k^λ and M_k^ρ are finite von Neumann algebras (equipped with faithful, normal, tracial states t_k^λ and t_k^ρ) that are commutants of each other and H_k can be identified as the standard module for both, with Ω_k being a cyclic and separating trace-vector for both.

In order to get a tower of von Neumann algebras we introduce a little more terminology. By a compatible pair of finite pre-von Neumann algebras, we will mean a pair (A, t_A) and (B, t_B) of finite pre-von Neumann algebras such that $A \subseteq B$ and $t_B|_A = t_A$. In particular, for any $k \geq 0$, $F_k(P) \subseteq F_{k+1}(P)$ equipped with their natural traces t_k and t_{k+1} give examples. Given such a pair of compatible pre-von Neumann algebras, identify H_A with a subspace of H_B so that $\Omega_A = \Omega_B = \Omega$, say.

We will need the following lemma which is an easy consequence of Theorem II.2.6 and Proposition III.3.12 of [Tks].

Lemma 4.5. *Suppose that $M \subseteq \mathcal{L}(H)$ and $N \subseteq \mathcal{L}(K)$ are von Neumann algebras and $\theta : M \to N$ is a $*$-homomorphism such that if $x_i \to x$ is a norm bounded strongly convergent net in M, then $\theta(x_i) \to \theta(x)$ strongly in N. Then θ is a normal map and its image is a von Neumann subalgebra of N.* $\square$

Proposition 4.6. *Let (A, t_A) and (B, t_B) be a compatible pair of finite pre-von Neumann algebras and Ω be as above. Let $\lambda_A : A \to \mathcal{L}(H_A)$ and $\lambda_B : B \to \mathcal{L}(H_B)$*

be the left regular representations of A and B respectively and let $M_A^\lambda = \lambda_A(A)''$ and $M_B^\lambda = \lambda_B(B)''$. Then,

(1) *The inclusion $A \subseteq B$ extends uniquely to a normal inclusion, say ι, of M_A^λ into M_B^λ with image $\lambda_B(A)''$ (where $A \subseteq M_A^\lambda$ by identification with $\lambda_A(A)$).*
(2) *For any $a'' \in M_A^\lambda$, $a''\Omega = \iota(a'')\Omega$, and in particular, $M_A^\lambda\Omega \subseteq M_B^\lambda\Omega$.*

Proof. (1) The subspace H_A of H_B is stable for $\lambda_B(A)$ and hence also for its double commutant. Since $\Omega \in H_A$ is separating for M_B^λ and hence also for $\lambda_B(A)''$, it follows that the map of compression to H_A is an injective $*$-homomorphism from $\lambda_B(A)''$ with image contained in $\lambda_A(A)'' = M_A^\lambda$. This map is clearly strongly continuous and so by Lemma 4.5 it is normal and its image which is a von Neumann algebra containing $\lambda_A(A)$ must be M_A^λ. Just let ι be the inverse map.
(2) Since the compression to H_A of $\iota(a'')$ is a'', $\Omega \in H_A$ and H_A is stable for $\lambda_B(A)''$, this is immediate. $\square$

Remark 4.7. It can be verified that ι is the map defined by $\iota(x)(b\Omega) = J_B(b^*x^*\Omega)$ for $x \in M_A^\lambda$ and $b \in B$.

Applying Proposition 4.6 to the tower $F_0(P) \subseteq F_1(P) \subseteq F_2(P) \subseteq \cdots$, each pair of successive terms of which is a compatible pair of pre-von Neumann algebras, we finally have a tower $M_0^\lambda \subseteq M_1^\lambda \subseteq \cdots$ of finite von Neumann algebras. Nevertheless we continue to regard M_k^λ as a subset of $\mathcal{L}(H_k)$ and note that the M_k^λ have a common cyclic and separating vector $H_0 \ni \Omega = \Omega_1 = \Omega_2 = \cdots$.

5. Factoriality and relative commutants

In this section we show that the tower $M_0^\lambda \subseteq M_1^\lambda \subseteq \cdots$ of finite von Neumann algebras constructed in Section 4 is in fact a tower of II_1-factors, and more generally that $(M_0^\lambda)' \cap M_k^\lambda$ is $P_k \subseteq F_k(P) \subseteq M_k^\lambda$. We begin by justifying the words 'more generally' of the previous sentence. Throughout this section, we fix a $k \geq 0$.

Lemma 5.1. *Suppose that $(M_0^\lambda)' \cap M_k^\lambda$ is $P_k \subseteq F_k(P) \subseteq M_k^\lambda$. Then, for $1 \leq i \leq k$, the relative commutant $(M_i^\lambda)' \cap M_k^\lambda = P_{i,k} \subseteq P_k$ (where $P_{i,k}$ is as in the last paragraph of Section 2). In particular, M_k^λ is a factor.*

Proof. A straightforward pictorial argument shows that $P_{i,k} \subseteq P_k \subseteq F_k(P)$ commutes with (the image in $F_k(P)$ of) $F_i(P)$ and hence also with (the image in M_k^λ of) M_i^λ (which is its double commutant by Proposition 4.6); it now suffices to see than an element of $P_k \subseteq F_k(P)$ that commutes with (the image in $F_k(P)$ of) $F_i(P)$ is necessarily in $P_{i,k}$. Take such an element, say $x \in P_k$ and consider the element of $P_{k+i} \subseteq F_i(P)$ shown on the left in Figure 11 which is seen to map (under the inclusion map) to the element of $P_{2k} \subseteq F_k(P)$ shown on the right.

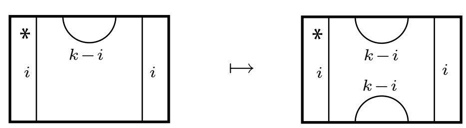
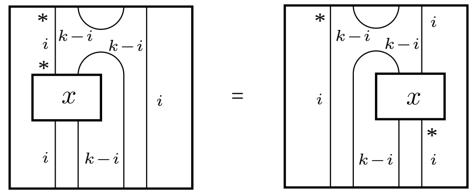


Figure 11. An element of $P_{k+i} \subseteq F_i(P)$ and its image in $P_{2k} \subseteq F_k(P)$

The condition that this element commutes with x in $F_k(P)$ is easily seen to translate to the tangle equation:



which holds in P_{2k}. But now, taking the conditional expectation of both sides into P_k shows that $\delta^i x = Z_{EL(i)^k_k}(x)$, whence $x \in P_{i,k}$. $\square$

Verification of the hypothesis of Lemma 5.1 is computationally involved and is the main result of this section which we state as a proposition. The bulk of the work in proving this proposition is contained in establishing Propositions 5.4 and 5.5. In the course of proving this proposition, the following notation and fact will be used. For $a \in F_k(P)$, define $[a] = \{\xi \in H_k : \lambda_k(a)(\xi) = \rho_k(a)(\xi)\}$, which is a closed subspace of H_k. Now observe that since Ω is separating for M_k^λ, for $x \in M_k^\lambda$, the operator equation $ax = xa$ is equivalent to the condition $x\Omega \in [a]$.

Proposition 5.2. $(M_0^\lambda)' \cap M_k^\lambda = P_k \subseteq F_k(P) \subseteq M_k^\lambda$.

Proof. As in the proof of Lemma 5.1, an easy pictorial calculation shows that $P_k \subseteq F_k(P)$ certainly commutes with all elements of (the image in $F_k(P)$ of) $F_0(P)$ and therefore also with (the image in M_k^λ of) M_0^λ (which is its double commutant by Proposition 4.6).

To verify the other containment, we will show that any element of M_k^λ that commutes with the specific elements $c, d \in F_0(P)$ shown in Figure 12 is necessarily

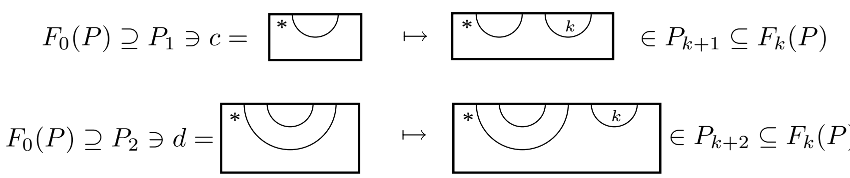


FIGURE 12. The elements $c, d \in F_0(P)$ and their images in $F_k(P)$

in $P_k \subseteq F_k(P)$. We remark that the elements c and d also play a prominent role in the proof of [GnnJnsShl].

Suppose now that $x \in M_k^\lambda$ commutes with both $c, d \in F_k(P)$, or equivalently that $x\Omega = (x_k, x_{k+1}, \cdots) \in [c] \cap [d]$. It then follows from Propositions 5.4 and 5.5 that $x\Omega = x_k\Omega$ or that $x = x_k \in P_k$. $\square$

Proposition 5.4 identifies $[c]$ as a certain explicitly defined (closed) subspace of $C_k \subseteq H_k$. Here $C_k = \oplus_{n=k}^\infty C_k^n$ where $C_k^n \subseteq P_n$ is defined as the range of $Z_{X_k^n}$ where X_k^n, defined for $n \geq k$, is the annular tangle in Figure 13, which has $n - k$ cups. We will have occasion to use the following observation.

Remark 5.3. We shall write $(C_k^n)^\perp$ to denote the orthogonal complement of C_k^n in P_n. It is a consequence of the faithfulness of τ that $x \in C_k^n$ exactly when it satisfies the capping condition of Figure 14.

Proposition 5.4. $[c] = C_k$.

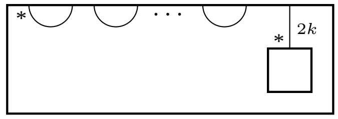


FIGURE 13. The tangle X_k^n

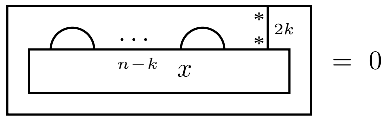


FIGURE 14. The capping condition

We will take up the proof of Proposition 5.4 after that of Proposition 5.5.

Proposition 5.5. $C_k \cap [d] = P_k \subseteq H_k$.

Proof. Suppose that $\xi = (x_k, x_{k+1}, \cdots) \in C_k$, so that for $n \geq k$, there exist $y_k^n \in P_k$ such that $x_n = Z_{X_k^n}(y_k^n)$. We then compute

$$\begin{aligned}
\lambda_k(d)(\xi) - \rho_k(d)(\xi) &= \sum_{n=k}^{\infty}(d\#x_n - x_n\#d) \\
&= \sum_{n=k}^{\infty} \sum_{t=|n-(k+2)|+k}^{n+2} (d\#x_n - x_n\#d)_t \\
&= \sum_{t=k}^{\infty} \sum_{n=|t-(k+2)|+k}^{t+2} [d, Z_{X_k^n}(y_k^n)]_t
\end{aligned}$$

Notice that if $t > k+2$, then the P_t component of $\lambda_k(d)(\xi) - \rho_k(d)(\xi)$ is given by

$$\sum_{n=t-2}^{t-1} [d, Z_{X_k^n}(y_k^n)]_t$$

since the other 3 commutators vanish, and is consequently given by $Z_{Y_k^t}(y_k^{t-1} + y_k^{t-2}) - Z_{Z_k^t}(y_k^{t-1} + y_k^{t-2})$ where Y_k^t and Z_k^t, defined for $t \geq k+2$, are the tangles in Figure 15, each having a double cup and $t-k-2$ single cups. Thus if $\xi \in [d]$,

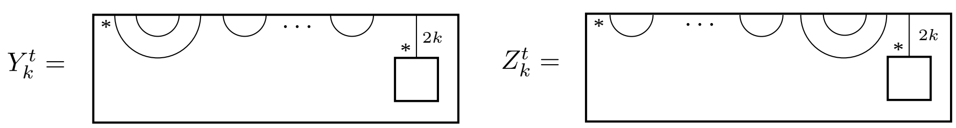


FIGURE 15. The tangles Y_k^t and Z_k^t

then for $t > k+2$, $Z_{Y_k^t}(y_k^{t-1} + y_k^{t-2}) = Z_{Z_k^t}(y_k^{t-1} + y_k^{t-2})$.

Note now that if there are at least 2 single cups, i.e., if $t \geq k+4$, then, capping off the points $\{4, 5\}$ of Y_k^t and Z_k^t gives Y_k^{t-1} and Z_k^{t-1}. It follows by induction that for $t \geq k+3$, $Z_{Y_k^{k+3}}(y_k^{t-1} + y_k^{t-2}) = Z_{Z_k^{k+3}}(y_k^{t-1} + y_k^{t-2})$. But now, capping off

$\{1,2\}$, $\{3,6\}$ and $\{4,5\}$ of Y_k^{k+3} and Z_k^{k+3} gives δ times the identity tangle I_k^k for Y_k^{k+3} and δ^3 times I_k^k for Z_k^{k+3} for Z_k^{k+3}. Since $\delta > 1$, it follows that for $t \geq k+3$, $y_k^{t-1} + y_k^{t-2} = 0$. Setting $y_{k+1} = y$, we have $y_{k+3} = y_{k+5} = \cdots = y = -y_{k+2} = -y_{k+4} = \cdots$.

Finally, since $x_n = Z_{X_k^n}(y_k^n)$, we see that $\tau(x_n^* x_n) = \tau((y_k^n)^* y_k^n) = \tau(y^* y)$ for $n \geq k+1$. Hence $||\xi||_{H_k}^2 = \tau(x_k^* x_k) + \tau(y^* y)(\delta + \delta^2 + \cdots)$. Since this is to be finite (and $\delta > 1$) , it follows that $y = 0$ or equivalently that $x_n = 0$ for $n \geq k+1$, and hence $\xi \in P_k \subseteq H_k$. □

The proof of Proposition 5.4 involves analysis of linear equations involving a certain class of annular tangles that we will now define. Given $m, n \geq k \geq 0$ and subsets $A \subseteq \{1, 2, \cdots, m-k\}$ and $B \subseteq \{1, 2, \cdots, n-k\}$ of equal cardinality we define an annular tangle $T(k, A, B)_n^m$ by the following three requirements: (i) there is a strictly increasing bijection $f_{AB} : A \to B$ such that for $\alpha \in A$, the marked points on the external box that are labelled by $2\alpha - 1$ and 2α are joined to the points on the internal box labelled by $2f_{AB}(\alpha) - 1$ and $2f_{AB}(\alpha)$, (ii) the last $2k$ points on the external box are joined with the last $2k$ points on the internal box, and (iii) the rest of the marked points on the internal and external boxes are capped off in pairs without nesting by joining each odd point to the next even point. (In all cases of interest, the sets A and B will be intervals of positive integers; for example we write $A = [3, 7]$ to mean $A = \{3, 4, 5, 6, 7\}$.) We illustrate with an example of the tangle $T(1, [4, 5], [3, 4])_5^8$ in Figure 16.

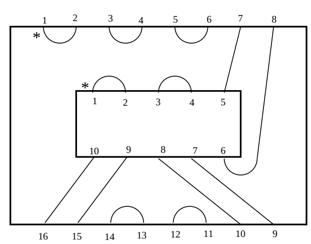


FIGURE 16. The tangle $T(1, [4, 5], [3, 4])_5^8$

We leave it to the reader to verify that the composition formula for this class of tangles is given by:

$$Z_{T(k,A,B)_n^m} \circ Z_{T(k,C,D)_p^n} = \delta^{n-k-|B \cup C|} Z_{T(k,E,F)_p^m},$$

where $E = f_{AB}^{-1}(B \cap C)$ and $F = f_{CD}(B \cap C)$.

We now isolate a key lemma used in the proof of Proposition 5.4.

Lemma 5.6. *For $n \geq k$, the map $(C_k^n)^\perp \ni x \mapsto z = (c\#x - x\#c)_{n+1} \in P_{n+1}$ is injective with inverse given by*

$$x = \sum_{t=1}^{n-k} \delta^{-t} Z_{T(k,[1,n+1-t-k],[t+1,n-k+1])_{n+1}^n}(z).$$

Proof. Pictorially, z is given in terms of x as in Figure 17. It then follows by applying appropriate annular tangles that for any t between 1 and $n-k$, the pictorial equation of Figure 18 holds (where the numbers on top of the ellipses

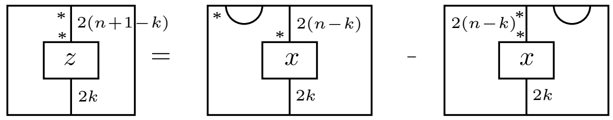


FIGURE 17. Pictorial expression for z in terms of x

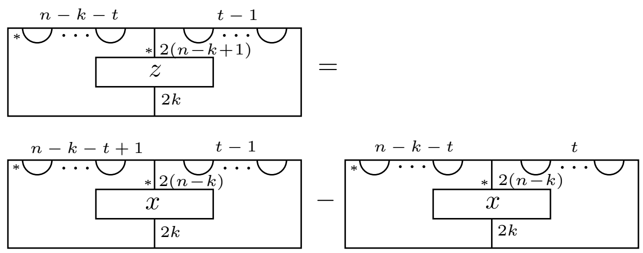


FIGURE 18. Consequence of Figure 17

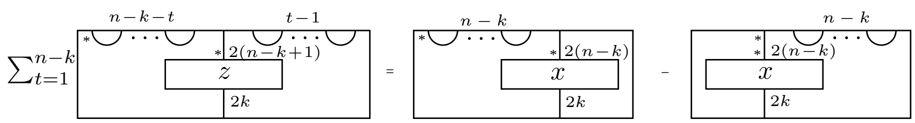


FIGURE 19. Consequence of Figure 18

indicate the number of cups). Sum over t between 1 and $n-k$ and use the patent telescoping on the right to get the equation of Figure 19.

Finally, cap off the pairs of points $\{1,2\},\{3,4\},\cdots,\{2(n-k)-1,2(n-k)\}$ on all the tangles in the Figure 19 to conclude that

$$\sum_{t=1}^{n-k}\delta^{n-k-t}Z_{T(k,[1,n+1-t-k],[t+1,n-k+1])^n_{n+1}}(z)=\delta^{n-k}x,$$

because the second term on the right vanishes by Remark 5.3. $\square$

Corollary 5.7. *Suppose that* $\xi=(x_k,x_{k+1},\cdots)\in\oplus_{n=k}^{\infty}(C_k^n)^{\perp}=C_k^{\perp}$ *and satisfies* $\lambda_k(c)(\xi)=\rho_k(c)(\xi)$. *Then, for* $m>n>k$ *with* $m-n=2d$, *we have:*

$$\begin{aligned} x_n &= \sum_{t=1}^{n-k}\delta^{-(t+d-1)}\left\{Z_{T(k,[1,n+1-t-k],[t+d,n-k+d])^n_m}(x_m)\right.\\ &\quad\left.-\,Z_{T(k,[1,n+1-t-k],[t+d+1,n-k+d+1])^n_m}(x_m)\right\}\end{aligned}$$

Proof. As in the proof of Proposition 5.5, some calculation shows that for $n>k$, the P_{n+1} component of $\lambda_k(c)(\xi)-\rho_k(c)(\xi)$ is seen to be

$$\sum_{s=n}^{n+2}[c,x_s]_{n+1}$$

of which the middle term is seen to vanish. Hence it is given by the difference of the right and left hand sides of the equation in Figure 20, and therefore if $\lambda_k(c)(\xi)=\rho_k(c)(\xi)$, the equations of Figure 20 hold for all $n>k$.

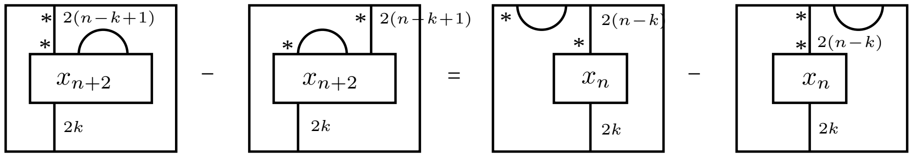


FIGURE 20. The condition for commuting with c

Let z denote the value of the left and right hand sides of the equation in Figure 20, so that on the one hand, we have

$$z = Z_{T(k,[1,n-k+1],[1,n-k+1])^{n+1}_{n+2}}(x_{n+2}) - Z_{T(k,[1,n-k+1],[2,n-k+2])^{n+1}_{n+2}}(x_{n+2}),$$

while on the other, $z = (c\#x_n - x_n\#c)_{n+1}$. We now have by Lemma 5.6 that

$$\begin{aligned} x_n &= \sum_{t=1}^{n-k} \delta^{-t} Z_{T(k,[1,n+1-t-k],[t+1,n-k+1])^{n}_{n+1}} \\ &\quad \left\{ Z_{T(k,[1,n-k+1],[1,n-k+1])^{n+1}_{n+2}}(x_{n+2}) - Z_{T(k,[1,n-k+1],[2,n-k+2])^{n+1}_{n+2}}(x_{n+2}) \right\} \\ &= \sum_{t=1}^{n-k} \delta^{-t} \left\{ Z_{T(k,[1,n+1-t-k],[t+1,n-k+1])^{n}_{n+2}}(x_{n+2}) \right. \\ &\quad \left. - Z_{T(k,[1,n+1-t-k],[t+2,n-k+2])^{n}_{n+2}}(x_{n+2}) \right\}, \end{aligned}$$

which proves the $d = 1$ case. For $d > 1$, assume by induction on d that

$$\begin{aligned} x_{n+2} = \sum_{s=1}^{n+2-k} \delta^{-(s+d-2)} \quad \times \quad & \left\{ Z_{T(k,[1,n+3-s-k],[s+d-1,n-k+d+1])^{n+2}_{m}}(x_m) \right. \\ & \left. - Z_{T(k,[1,n+3-s-k],[s+d,n-k+d+2])^{n+2}_{m}}(x_m) \right\} \end{aligned}$$

Substituting the expression for x_{n+2} from the last equation into the preceding equation, we find:

$$\begin{aligned} x_n &= \sum_{t=1}^{n-k} \sum_{s=1}^{n+2-k} \delta^{-(t+s+d-2)} \times \\ &\quad \left\{ Z_{T(k,[1,n+1-t-k],[t+1,n-k+1])^{n}_{n+2}} \circ Z_{T(k,[1,n+3-s-k],[s+d-1,n-k+d+1])^{n+2}_{m}}(x_m) \right. \\ &\quad - Z_{T(k,[1,n+1-t-k],[t+1,n-k+1])^{n}_{n+2}} \circ Z_{T(k,[1,n+3-s-k],[s+d,n-k+d+2])^{n+2}_{m}}(x_m) \\ &\quad - Z_{T(k,[1,n+1-t-k],[t+2,n-k+2])^{n}_{n+2}} \circ Z_{T(k,[1,n+3-s-k],[s+d-1,n-k+d+1])^{n+2}_{m}}(x_m) \\ &\quad \left. + Z_{T(k,[1,n+1-t-k],[t+2,n-k+2])^{n}_{n+2}} \circ Z_{T(k,[1,n+3-s-k],[s+d,n-k+d+2])^{n+2}_{m}}(x_m) \right\} \end{aligned}$$

This is a sum over varying t and s of 4 terms, each with a multiplicative factor of a power of δ. Note that as $x_m \in (C_k^m)^{\perp}$, it follows from Remark 5.3 that the first two terms vanish unless $t \leq n + 2 - s - k$ and the last two vanish unless $t \leq n + 1 - s - k$. So the sum of the first two terms may be taken over those (t, s) satisfying $t + s \leq n + 2 - k$ while the sum of the last two terms may be taken over those (t, s) satisfying $t + s \leq n + 1 - k$. In this range, the composition formula for the $T(k, A, B)$ tangles gives:

$$
\begin{aligned}
Z_{T(k,[1,n+1-t-k],[t+1,n-k+1])^n_{n+2}} &\circ Z_{T(k,[1,n+3-s-k],[s+d-1,n-k+d+1])^{n+2}_m} \\
&= \begin{cases} Z_{T(k,[1,n+1-t-k],[t+d,n-k+d])^n_m} \text{ if } s=1 \\ \delta Z_{T(k,[1,n+3-s-k-t],[s+t+d-1,n-k+d+1])^n_m} \text{ if } s>1 \end{cases} \\
Z_{T(k,[1,n+1-t-k],[t+1,n-k+1])^n_{n+2}} &\circ Z_{T(k,[1,n+3-s-k],[s+d,n-k+d+2])^{n+2}_m} \\
&= \begin{cases} Z_{T(k,[1,n+1-t-k],[t+d+1,n-k+d+1])^n_m} \text{ if } s=1 \\ \delta Z_{T(k,[1,n+3-s-k-t],[s+t+d,n-k+d+2])^n_m} \text{ if } s>1 \end{cases} \\
Z_{T(k,[1,n+1-t-k],[t+2,n-k+2])^n_{n+2}} &\circ Z_{T(k,[1,n+3-s-k],[s+d-1,n-k+d+1])^{n+2}_m} \\
&= Z_{T(k,[1,n+2-s-k-t],[s+t+d,n-k+d+1])^n_m} \\
Z_{T(k,[1,n+1-t-k],[t+2,n-k+2])^n_{n+2}} &\circ Z_{T(k,[1,n+3-s-k],[s+d,n-k+d+2])^{n+2}_m} \\
&= Z_{T(k,[1,n+2-s-k-t],[s+t+d+1,n-k+d+2])^n_m}
\end{aligned}
$$

Finally, note that the first two terms with $s = 1$ sum to exactly the desired expression for x_n while for $s > 1$, the first term corresponding to (t, s) cancels against the third term corresponding to $(t, s-1)$ while the second term corresponding to (t, s) cancels with the fourth term corresponding to $(t, s-1)$, finishing the induction. $\square$

We only need one more thing in order to complete the proof of Proposition 5.4.

Lemma 5.8. *Suppose that $x = x_n \in P_n \subseteq F_k(P)$ and let $y = Z_{T(k,A,B)^m_n}(x) \in P_m \subseteq F_k(P)$ for some annular tangle $T(k, A, B)^m_n$. Then $||y||_{H_k} \leq \delta^{\frac{1}{2}(n+m)-(|A|+k)}||x||_{H_k}$.*

Proof. Since for $z \in P_u \subseteq F_k(P)$, the norm relation $\delta^{k-u}||z||^2_{H_k} = ||z||^2_{P_u} = \tau(z^*z)$ holds, the inequality we need to verify may be equivalently stated as $||y||^2_{P_m} \leq \delta^{2(n-|A|-k)}||x||^2_{P_n}$. We leave it to the reader to see that this follows from Remark 2.2. $\square$

Proof of Proposition 5.4. Observe first that C^n_k is easily verified to be in $[c]$ and so $C_k \subseteq [c]$. To show the other inclusion we will show that $C^\perp_k \cap [c] = \{0\}$.

Take $\xi = (x_k, x_{k+1}, \cdots) \in C^\perp_k = \oplus^\infty_{n=k}(C^n_k)^\perp$ and suppose that $\xi \in [c]$. Then, $x_k = 0$ since $C^k_k = P_k$ and $x_k \in (C^k_k)^\perp$. Next, it follows from Corollary 5.7 and Lemma 5.8 that if $m > n > k$ and $m - n = 2d$, then

$$||x_n||_{H_k} \leq \sum_{t=1}^{n-k} \delta^{-(t+d-1)}\delta^{\frac{1}{2}(n+(n+2d))-(n+1-t)}2||x_m||_{H_k} = 2(n-k)||x_m||_{H_k}.$$

Since $\xi \in H_k$ clearly implies that $lim_{m\to\infty}||x_m|| = 0$, it follows that $x_n = 0$ also for $n > k$. Hence $\xi = 0$. $\square$

We will refer to $M^\lambda_0 \subseteq M^\lambda_1$ as the subfactor constructed from P.

6. Basic construction tower, Jones projections and the main theorem

In this section, we first show that the tower $M^\lambda_0 \subseteq M^\lambda_1 \subseteq \cdots$ of II_1-factors may be identified with the basic construction tower of $M^\lambda_0 \subseteq M^\lambda_1$ which is extremal and has index δ^2. We also explicitly identify the Jones projections. Using this, we easily prove our main theorem.

We begin with an omnibus proposition.

Proposition 6.1. *For $k \geq 1$, the following statements hold.*

(1) *The trace preserving conditional expectation map $E^{\lambda}_{k-1} : M^{\lambda}_k \to M^{\lambda}_{k-1}$ is given by the restriction to M^{λ}_k of the continuous extension $H_k \to H_{k-1}$ of $E_{k-1} : F_k(P) \to F_{k-1}(P)$ (where $M^{\lambda}_k \subseteq H_k$ by $x \mapsto x\Omega$). It is continuous for the strong operator topologies on the domain and range.*

(2) *The element $e_{k+1} \in P_{k+1} \subseteq F_{k+1}(P)$ commutes with $F_{k-1}(P)$ and satisfies $E^{\lambda}_k(e_{k+1}) = \delta^{-2}$ and $e_{k+1}xe_{k+1} = E^{\lambda}_{k-1}(x)e_{k+1}$, for any $x \in M^{\lambda}_k$.*

(3) *The map $\theta_k : F_{k+1}(P) \to End(F_k(P))$ defined by $\theta_k(a)(b) = \delta^2 E_k(abe_{k+1})$ is a homomorphism.*

(4) *The tower $M^{\lambda}_{k-1} \subseteq M^{\lambda}_k \subseteq M^{\lambda}_{k+1}$ of II_1-factors is (isomorphic to) a basic construction tower with Jones projection given by e_{k+1} and finite index δ^2.*

(5) *The subfactor $M^{\lambda}_{k-1} \subseteq M^{\lambda}_k$ is extremal.*

Proof. (1) Since Lemma 4.4(4) identifies H_k with $L^2(M^{\lambda}_k, t^{\lambda}_k)$, it suffices to see that if $\tilde{e}_{k+1} \in \mathcal{L}(H_k)$ (for $k \geq 1$) denotes the orthogonal projection onto the closed subspace H_{k-1}, then $\tilde{e}_{k+1}|_{F_k(P)} = E_{k-1}$. Since $H_k = \oplus_{n=k}^{\infty} P_n$ while H_{k-1} is the sum of the subspaces $\oplus_{n=k}^{\infty} P_{n-1}$, it suffices to see that the restriction of E_{k-1} to $P_n \subseteq F_k(P)$ is the orthogonal projection of P_n onto P_{n-1} included in P_n via the specification of Figure 5, which is easily verified. Continuity for the strong operator topologies follows since $E^{\lambda}_{k-1}(x)$ is the compression of x to the subspace H_{k-1}.
(2) We omit the simple calculations needed to verify this.
(3) Denote $\theta_k(a)(b)$ by $a.b$. What is to be seen then, is that $(a\#b).c = a.(b.c)$ for $a, b \in F_{k+1}(P)$ and $c \in F_k(P)$. Some computation with the definitions yields for $a \in P_m \subseteq F_{k+1}(P)$ and $b \in P_n \subseteq F_k(P)$, $a.b = \sum_{t=|m-n-1|+k}^{m+n-k-1} (a.b)_t$ where $(a.b)_t$ is given by the tangle in Figure 21. Thus for $a \in P_m \subseteq F_{k+1}(P), b \in P_n \subseteq F_{k+1}(P)$

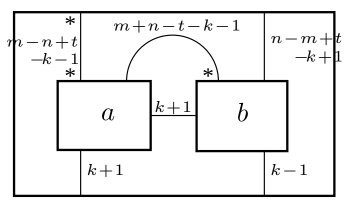


FIGURE 21. Definition of the P_t component of $a.b$.

and $c \in P_p \subseteq F_k(P)$, we have:

$$(a\#b).c = \sum_{(t,s)\in I(m,n,p)} ((a\#b)_t.c)_s \quad \text{and} \quad a.(b.c) = \sum_{(v,u)\in J(m,n,p)} (a.(b.c)_v)_u,$$

where $I(m,n,p) = \{(t,s) : |m-n|+k+1 \leq t \leq m+n-k-1, |t-p-1|+k \leq s \leq t+p-k-1\}$ and $J(m,n,p) = \{(v,u) : |n-p-1|+k \leq v \leq n+p-k-1, |m-v-1|+k \leq u \leq m+v-k-1\}$. We leave it to the reader to check that (a) $I(m,n,p) = J(p+1,n,m-1)$ and (b) the map $T(m,n,p) : I(m,n,p) \to J(m,n,p)$ defined by $(t,s) \mapsto (v,u) = (max\{m+p, n+s\}-t, s)$ is a well-defined bijection with inverse $T(p+1,n,m-1)$ such that (c) if $(t,s) \mapsto (v,u)$ then both $((a\#b)_t.c)_s$ and $(a.(b.c)_v)_u$ are equal to the figure on the left or on the right in Figure 22 according as $m+p \geq n+s$ or $m+p \leq n+s$.
(4) Extend θ_k to a map, also denoted $\theta_k : M^{\lambda}_{k+1} \to \mathcal{L}(H_k)$ by defining it on

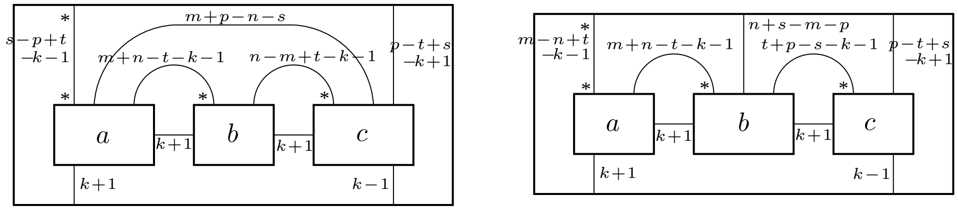


FIGURE 22. $((a\#b)_t.c)_s = (a.(b.c)_v)_u$.

the dense subspace $M_k^\lambda\Omega \subseteq H_k$ by the formula (from Lemma 4.3.1 of [JnsSnd]) $\theta_k(x)(y\Omega) = \delta^2 E_k^\lambda(xye_{k+1})$ for $x \in M_{k+1}^\lambda$ and $y \in M_k^\lambda$ and observing that it extends continuously to H_k. It is easy to check that θ_k is $*$-preserving and it follows from the strong continuity of E_k^λ that θ_k preserves norm bounded strongly convergent nets. Further, by (3), restricted to $F_{k+1}(P)$, the map θ_k is a unital $*$-homomorphism. Now Kaplansky density, the convergence preserving property of θ_k and the strong continuity of multiplication on norm bounded subsets imply that θ_k is a unital $*$-homomorphism. Since its domain is a factor, θ_k is also injective. By Lemma 4.5, θ_k is normal so that its image is a von Neumann subalgebra of $\mathcal{L}(H_k)$.

That this image contains $M_k^\lambda = \theta_k(M_k^\lambda)$ and $\tilde{e}_{k+1} = \theta_k(e_{k+1})$ is clear from the formula that defines θ_k. Therefore (with $J_k : H_k \to H_k$ denoting the modular conjugation operator for M_k^λ) $J_k.im(\theta_k).J_k$ contains M_k^ρ and $\tilde{e}_{k+1}$. Direct calculation also shows that for $x \in M_{k+1}^\lambda$ and $y \in M_k^\lambda$, $J_k\theta_k(x^*)J_k(y\Omega) = \delta^2 E_k^\lambda(e_{k+1}yx)$ and consequently that $J_k.im(\theta_k).J_k \subseteq \lambda_k(M_{k-1}^\lambda)'$. The opposite inclusion is equivalent to the statement that $J_k.im(\theta_k)'.J_k \subseteq \lambda_k(M_{k-1}^\lambda)$. But then, $J_k.im(\theta_k)'.J_k \subseteq M_k^\lambda \cap (\tilde{e}_{k+1})'$ which is easily verified to be $\lambda_k(M_{k-1}^\lambda)$ since $\tilde{e}_{k+1}$ implements the conditional expectation of M_k^λ onto M_{k-1}^λ. We conclude that $im(\theta_k)$ is $J_k.\lambda_k(M_{k-1}^\lambda)'.J_k$.

So the II_1-factor M_{k+1}^λ is isomorphic (via θ_k) to the basic construction of $M_{k-1}^\lambda \subseteq M_k^\lambda$ with the Jones projection being identified with $e_{k+1} \in M_{k+1}^\lambda$. It also follows that the index is finite and equals $E_k^\lambda(e_{k+1})^{-1} = \delta^2$.

(5) By (4) it suffices to see that $M_0^\lambda \subseteq M_1^\lambda$ is extremal for which, according to Corollary 4.5 of [PmsPpa], it is enough that the anti-isomorphism $x \mapsto Jx^*J$ from $(M_0^\lambda)' \cap M_1^\lambda$ to $M_2^\lambda \cap (M_1^\lambda)'$ be trace preserving, for the trace t_1^λ on the left and the trace t_2^λ on the right. Here, all algebras involved are identified with subalgebras of $\mathcal{L}(H_1)$ via the appropriate isomorphisms - ι for M_0^λ and θ_1 for M_2^λ. By Lemma 5.1, $(M_0^\lambda)' \cap M_1^\lambda$ is identified with $P_1 \subseteq F_1(P) \subseteq \mathcal{L}(H_1)$ and $M_2^\lambda \cap (M_1^\lambda)'$ with $P_{1,2} \subseteq P_2 \subseteq F_2(P)$.

Working with the definition of θ_1 shows that if $x \in P_1$ and $z \in P_{1,2}$ are as in Figure 23 and $y \in F_1(P)$ is arbitrary, then $\theta_1(z)(y\Omega) = yx\Omega = Jx^*J(y\Omega)$. Chasing through the identifications, this establishes that that the anti-isomorphism $P_1 \to P_{1,2}$ given by $x \mapsto Jx^*J$ is the obvious one in Figure 23. Finally, noting that

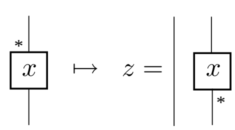


FIGURE 23. The map from P_1 to $P_{1,2}$

the traces restricted to P_1 and $P_{1,2}$ are both τ, sphericality of the subfactor planar algebra P proves the extremality desired. □

We now have all the pieces in place to prove our main theorem which is exactly that of [GnnJnsShl].

Theorem 6.2. *Let P be a subfactor planar algebra. The subfactor $M_0^\lambda \subseteq M_1^\lambda$ constructed from P is a finite index and extremal subfactor with planar algebra isomorphic to P.*

Proof. By Proposition 6.1 and the remarks preceding it, the subfactor $M_0^\lambda \subseteq M_1^\lambda$ constructed from P is extremal, of index δ^2 (where δ is the modulus of P) and has basic construction tower given by $M_0^\lambda \subseteq M_1^\lambda \subseteq M_2^\lambda \subseteq \cdots$.

Further, the towers of relative commutants $(M_1^\lambda)' \cap M_k^\lambda \subseteq (M_0^\lambda)' \cap M_k^\lambda$ are identified, by Proposition 5.2, with those of the $P_{1,k} \subseteq P_k$ with inclusions (as k increases) given by the inclusion tangle, by the remarks preceding Proposition 3.1.

Under this identification, the Jones projections $e_{k+1} \in P_{k+1}$ are the Jones projections in the basic construction tower $M_0^\lambda \subseteq M_1^\lambda \subseteq M_2^\lambda \subseteq \cdots$. Also the trace t_k^λ restricted to P_k is just its usual trace τ. It follows that the trace preserving conditional expectation maps $(M_0^\lambda)' \cap M_k^\lambda \to (M_0^\lambda)' \cap M_{k-1}^\lambda$ and $(M_0^\lambda)' \cap M_k^\lambda \to (M_1^\lambda)' \cap M_k^\lambda$ agree with those of the planar algebra P.

An appeal now to Jones' theorem (Theorem 2.1) completes the proof. □

7. Agreement with the Guionnet-Jones-Shlyakhtenko model

In this short section, we construct (without proofs) explicit trace preserving $*$-isomorphisms from the graded algebras $Gr_k(P)$ of [GnnJnsShl] to the filtered algebras $F_k(P)$ and their inverse isomorphisms. Since we have not unravelled the inclusions of $Gr_k(P)$ from [GnnJnsShl], we are not able to claim the towers are isomorphic, but suspect that this is indeed the case.

For $k \geq 0$, the algebra $Gr_k(P)$ is defined to be the graded algebra with underlying vector space given by $\oplus_{n=k}^{\infty} P_n$ and multiplication defined as follows. For $a \in P_m \subseteq Gr_k(P)$ and $b \in P_n \subseteq Gr_k(P)$, define $a \bullet b \in P_{m+n-k} \subseteq Gr_k(P)$ by $a \bullet b = (a\#b)_{m+n-k}$ - the highest (i.e., $t = m+n-k$) degree component of $a\#b$ (and observe therefore that $Gr_k(P)$ is the associated graded algebra of the filtered algebra $F_k(P)$).

There is a $*$-algebra structure on $Gr_k(P)$ defined exactly as in $F_k(P)$ which also we will denote by †. There is also a trace Tr_k defined on $Gr_k(P)$ as follows. For $a \in P_m \subseteq Gr_k(P)$, define $Tr_k(a)$ by the tangle in Figure 24 where $T_m \in P_m$ is

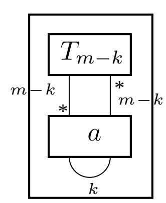


Figure 24. Definition of $Tr_k(a)$.

defined to be the sum of all the Temperley-Lieb elements of P_m.

Define maps $\phi(k) : Gr_k(P) \to F_k(P)$ and $\psi(k) : F_k(P) \to Gr_k(P)$ as follows. For $i, j \geq k$, denoting the P_i component of $\phi(k)|_{P_j}$ (resp. $\psi(k)|_{P_j}$) by $\phi(k)^i_j$ (resp. $\psi(k)^i_j$), define $\phi(k)^i_j$ to be the sum of the maps given by all the 'k-good (j, i)-annular tangles' and $\psi(k)^i_j$ to be $(-1)^{i+j}$ times the sum of the maps given by all the 'k-excellent (j, i)-annular tangles', where, for $k \leq i \leq j$, a (j, i)-annular tangle is said to be k-good if (i) it has $2i$ through strands, (ii) the $*$-regions of its internal and external boxes coincide and (iii) the last $2k$ points of the internal box are connected to the last $2k$ points of the external box, and k-excellent if further, (iv) there is no nesting among the strands connecting points on its internal box. Note that the matrices of $\phi(k)$ and $\psi(k)$, regarded as maps from $\oplus_{n=k}^{\infty} P_n$ to itself, are block upper triangular with diagonal blocks being identity matrices, and consequently clearly invertible.

We then have the following result.

Proposition 7.1. *For each $k \geq 0$, the maps $\phi(k)$ and $\psi(k)$ are mutually inverse $*$-isomorphisms which take the traces Tr_k and $\delta^k t_k$ to each other.*

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