ON THE GUIONNET-JONES-SHLYAKHTENKO CONSTRUCTION FOR GRAPHS

VIJAY KODIYALAM AND V. S. SUNDER

Abstract. Using an analogue of the Guionnet-Jones-Shlaykhtenko construction for graphs we show that their construction applied to any subfactor planar algebra of finite depth yields an inclusion of interpolated free group factors with finite parameter, thereby giving another proof of their universality for finite depth planar algebras.

The main theorem of [GnnJnsShl2008] constructs an extremal finite index II_1 subfactor $N = M_0 \subseteq M_1 = M$ from a subfactor planar algebra P with the property that the planar algebra of $N \subseteq M$ is isomorphic to P. We show in this paper that if P is a subfactor planar algebra of modulus $\delta > 1$ and of finite depth, then, for the associated subfactor $N \subseteq M$, there are isomorphisms $N \cong LF(r)$ and $M \cong LF(s)$ for some $1 < r, s < \infty$, where $LF(t)$ for $1 < t < \infty$ is the interpolated free group factor of [Dyk1994] and [Rdl1994]. This can be regarded as yet another proof of the fact - see [Rdl1994] and [Dyk2002] - that interpolated free group factors with finite parameter are universal for finite depth subfactor planar algebras. The word 'universal' above is used in the sense of [PpaShl2003] where they essentially prove that $LF(\infty)$ is universal for all subfactor planar algebras.

We shall now outline the structure of this paper. In §1 we construct - see Proposition 1 - a graded, tracial, faithful $*$-probability space $Gr(\Gamma)$ associated to a finite, weighted, bipartite graph Γ and establish - see Proposition 5 - an isomorphism between $Gr(\Gamma)$ and a filtered, tracial, faithful $*$-probability space $F(\Gamma)$ - see Proposition 4. Our main interest will be in an associated finite von Neumann algebra $M(\Gamma)$ and some of its corners determined by sets of vertices of Γ - specifically the corner $M(\Gamma, 0)$ (respectively $M(\Gamma, 1)$) determined by the set of even (respectively odd) vertices of Γ. The main result in §2 asserts - see Theorem 21 - that if Γ is a connected graph with more than one edge, then, $M(\Gamma)$ is the direct sum of a II_1 factor and a finite-dimensional abelian algebra. The goal of §3 is to express $Gr(\Gamma, 0)$ and $M(\Gamma, 0)$ - see Proposition 26 and equation (23) - as amalgamated free products of the corresponding algebras associated to subgraphs with a single odd vertex. In §4 we determine the structure of $M(\Lambda, 0)$ - see Corollary 33 - for a graph Λ with a single odd vertex. The penultimate §5 proves - see Theorem 35 - one of our main results : for a connected graph Γ with more than one edge and equipped with its Perron-Frobenius weighting, the algebra $M(\Gamma)$ is an interpolated free group factor with finite parameter. The final §6 applies this - see Theorems 41 and 42 - to show that the Guionnet-Jones-Shlyakhtenko (henceforth GJS) construction applied to a finite depth subfactor planar algebra yields an inclusion of interpolated free group factors with finite parameters.

1. The global graded probability space associated to a graph

The goal of this section is to associate a graded, tracial, faithful $*$-probability space $Gr(\Gamma)$ and a von Neumann algebra $M(\Gamma)$ to a graph Γ . Recall that a tracial $*$-probability space consists of a unital, complex $*$-algebra A equipped with a trace $\tau : A \to \mathbb{C}$ that satisfies $\tau(1) = 1$ and $\tau(a^*a) \geq 0$, for all $a \in A$. It is said to be graded if the algebra A is graded and to be faithful if $\tau(a^*a) = 0 \Rightarrow a = 0$.

Throughout this paper, by a graph, we will mean a finite, weighted, bipartite graph which consists of the following data: (i) a finite set V of 'vertices' partitioned as $V_0 \coprod V_1$ - the sets V_0 and V_1 will be referred to as sets of even and odd vertices respectively, (ii) a finite set E of 'edges' equipped with 'start' and 'finish' maps $s, f : E \to V$ and a 'reversal' involution $\xi \mapsto \tilde{\xi}$ of E intertwining s and f such that $s(\xi) \in V_0 \Leftrightarrow f(\xi) \in V_1$, and (iii) a 'weighting' which is a function $\mu : V \to \mathbb{R}_+$ normalised such that $\sum_{v \in V} \mu^2(v) = 1$.

For us, the main examples of such graphs are the principal graphs of non-trivial II_1-subfactors of finite depth (where μ is given by the square root of an appropriately normalised Perron-Frobenius eigenvector) and their subgraphs (with the restricted μ appropriately normalised).

The construction of $Gr(\Gamma)$ involves paths in Γ, notations and definitions for which we discuss briefly. A path ξ in Γ is denoted

$$(v_0^\xi \stackrel{\xi_1}{\to} v_1^\xi \stackrel{\xi_2}{\to} v_2^\xi \stackrel{\xi_3}{\to} \cdots \stackrel{\xi_n}{\to} v_n^\xi),$$

where $v_i^\xi \in V$ and $(v_{i-1}^\xi \stackrel{\xi_i}{\to} v_i^\xi) \in E$, with the notation being self-explanatory. The start and finish vertex functions on paths in Γ will also be denoted by $s(\cdot)$ and $f(\cdot)$ respectively and the length function by $\ell(\cdot)$, so that $s(\xi) = v_0^\xi, f(\xi) = v_n^\xi$ and $\ell(\xi) = n$. For $0 \leq i \leq j \leq n$, we will use notation such as $\xi_{[i,j]}$ for the path $(v_i^\xi \stackrel{\xi_{i+1}}{\to} v_{i+1}^\xi \stackrel{\xi_{i+2}}{\to} v_{i+2}^\xi \cdots \stackrel{\xi_j}{\to} v_j^\xi)$, where the interval refers to the vertex indices. The symbol $\circ$ will denote composition of paths and $\tilde{}$ will stand for path reversal. For $n \geq 0$, the path space $P_n(\Gamma)$ associated to the graph Γ is the complex vector space with basis $\{[\xi] : \xi \text{ is a path of length } n \text{ in } \Gamma\}$.

We will now define $Gr(\Gamma)$ and its structure maps. As a graded vector space, $Gr(\Gamma) = \oplus_{n\geq 0} P_n(\Gamma)$. The multiplication in $Gr(\Gamma)$, denoted by $\bullet$, is given by concatenation on the path basis and extended by linearity:

$$[\xi] \bullet [\eta] = \begin{cases} 0 & \text{if } f(\xi) \neq s(\eta) \\ [\xi \circ \eta] & \text{if } f(\xi) = s(\eta). \end{cases}$$

The involution $*$ on $Gr(\Gamma)$ is defined by conjugate linear extension of the reversal map $\tilde{}$ on the path basis, i.e., $[\xi]^* = [\tilde{\xi}]$. We define a linear functional τ on $Gr(\Gamma)$ motivated by the GJS trace. Suppose that $[\xi] \in P_n(\Gamma)$ for $n \geq 1$. Define

$$\tau([\xi]) = \sum_T \tau_T([\xi])$$

where the sum is over all Temperley-Lieb equivalence relations[1] T on $\{1, 2, \cdots, n\}$ (so that it is an empty sum, hence vanishes, for n odd) and τ_T is defined by

$$\tau_T([\xi]) = \prod_{\{\{i,j\}\in T : i<j\}} \delta_{\xi_i, \tilde{\xi_j}} \prod_{C \in K(T)} \mu(v^\xi_C)^{2-|C|}$$

where (i) $K(T)$ is the Kreweras complement of T - see [NcaSpc2006] - which is also a non-crossing partition of $\{1, 2, \cdots, n\}$ and (ii) $v^\xi_C = v^\xi_c$ for any $c \in C$ (all of which must be equal if the first product is non-zero). When $n = 0$, we set $\tau([(v)]) = \mu^2(v)$.

Proposition 1. *$Gr(\Gamma)$ is a graded, unital, associative, $*$-algebra and τ is a normalised trace on $Gr(\Gamma)$.*

Proof. The only not completely obvious assertion is the traciality of τ, which too follows, after a little thought, from the rotational invariance of the set of all TL-equivalence relations and from the definition of the product in $Gr(\Gamma)$. $\square$

Note that the multiplicative identity of $Gr(\Gamma)$ is the element $\sum_{v\in V} [(v)] \in P_0(\Gamma)$. In view of the fact that the different $[(v)]$, for $v \in V$, are orthogonal idempotents (adding to 1), we will denote $[(v)]$ also by e_v. It is useful to observe that an element, say x, of $P_n(\Gamma)$ may be regarded as the square matrix, with rows and columns indexed by V, with (v, w) entry given by $e_v x e_w$ (the part of x which is a linear combination of paths beginning at v and ending at w).

The proof of positivity and faithfulness of the trace τ involves some work with a different avatar of $Gr(\Gamma)$ which we will find very useful. We begin by recalling, from [JnsShlWlk2008], the category epi-TL which we will denote by $\mathcal{E}$. The objects of $\mathcal{E}$ are denoted $[n]$ for $n \geq 0$ and thought of as n-points (labelled $1, 2, 3, \cdots, n$) arranged on a horizontal line. A morphism in $Hom([n], [m])$ consists of a rectangle with m-points on the top horizontal line, n-points on the bottom horizontal line

[1] These are the non-crossing relations with every class having two elements.

and a Temperley-Lieb like tangle in between, subject to the restriction that each of the points above is joined to a point below. It must be observed that $Hom([n],[m])$ is non-empty precisely when $n-m$ is a non-negative even integer. Morphisms are composed by vertical stacking.

The morphisms in $\mathcal{E}$ are generated by those which have a single cap on the bottom line. Let $S_i^n : [n] \to [n-2]$ (for $1 \leq i < n$) denote the generator with the i^{th} and $(i+1)^{st}$ points on the bottom line capped. Some work shows that all relations among the morphisms are consequences of the relations

$$S_p^{n-2} S_q^n = S_q^{n-2} S_{p+2}^n \tag{1}$$

for $n-2 > p \geq q \geq 1$. In fact any element of $Hom([m+2k],[m])$ is uniquely expressible in the form $S_{i_1}^{m+2} S_{i_2}^{m+4} \cdots S_{i_k}^{m+2k}$ with $1 \leq i_1 < i_2 < \cdots < i_k < m+2k$. (The left end points of the k caps of the morphism are precisely at the places $i_1, i_2, \cdots, i_k$.) Such a morphism will be called non-nested if the caps are 'not nested', or equivalently, if $i_{j+1} \geq i_j + 2$ for each $j < k$ in its 'canonical decomposition' as above.

It follows that the category $\mathcal{E}$ 'acts' on the collection of vector spaces $P_n(\Gamma)$ in the sense that any element of $Hom([n],[m])$ yields a vector space homomorphism $P_n(\Gamma) \to P_m(\Gamma)$ with this assignment being compatible with compositions on both sides. Such an action can be defined[2] with S_i^n acting by

$$S_i^n([\xi]) = \delta_{\xi_i, \widetilde{\xi_{i+1}}} \frac{\mu(v_i^\xi)}{\mu(v_{i\pm 1}^\xi)} [\xi_{[0,i-1]} \circ \xi_{[i+1,n]}] \tag{2}$$

for $[\xi] \in P_n(\Gamma)$. More generally, given an arbitrary $S \in Hom([n],[m])$, it specifies a partition of $[n]$ as $T \cup E$, where T is the subset of points in $[n]$ that are joined to a point in $[m]$ and E is its complement. It also specifies a Temperley-Lieb equivalence relation $\sim$ on E. The action of S is then explicitly given by

$$S([\xi]) = \prod_{\{i,j\} \in \ \sim : i<j} \left(\delta_{\xi_i, \tilde{\xi_j}} \frac{\mu(v_i^\xi)}{\mu(v_j^\xi)} \right) [\circ_{t \in T} \xi_t], \tag{3}$$

where the concatenation is done in increasing order of elements of T and is interpreted as $[(f(\xi))]$ if $T = \emptyset$. (As in the equations displayed above, we shall often identify elements of $Hom([n],[m])$ with the associated operators from $P_n(\Gamma)$ to $P_m(\Gamma)$.)

The following lemma is a special case (of Proposition 3) which both motivates and is used in the proof of a different expression for $S([\xi])$ when $S \in Hom([2n],[0])$.

[2]Thus we are saying that the operators defined by equation (2) satisfy the relations (1).

Note that in this case, $E = \{1, 2, \cdots, 2n\}$ and $\sim = S$ regarded as an equivalence relation.

Lemma 2. *Let* $[\xi] \in P_{2n}(\Gamma)$ *and* $S \in Hom([2n], [0])$ *be given by* $\{\{1, 2n\}, \{2, 2n-1\}, \cdots, \{n, n+1\}\}$*. Then,*

$$S([\xi]) = \frac{\mu(v_n^\xi)}{\mu(v_{2n}^\xi)} \prod_{i=1}^{n} \delta_{\xi_i, \widetilde{\xi_{2n+1-i}}} \times [(v_{2n}^\xi)].$$

Proof. We may assume that ξ is a path consistent with S in the sense that $\xi_i = \widetilde{\xi_j}$ whenever $\{i, j\} \in S$, since otherwise, both sides of the desired equality vanish. Thus, $\xi_i = \widetilde{\xi_{2n+1-i}}$ and in particular, $v_i^\xi = v_{2n-i}^\xi$ for each $i = 0, 1, \cdots, 2n$.

Using equation (3), it now suffices to check that

$$\prod_{i=1}^{n} \frac{\mu(v_i^\xi)}{\mu(v_{2n+1-i}^\xi)} = \frac{\mu(v_n^\xi)}{\mu(v_{2n}^\xi)}.$$

But substituting $v_i^\xi = v_{2n-i}^\xi$, we see that the product on the left telescopes to the expression on the right. □

We next treat the case of a general S.

Proposition 3. *For* $[\xi] \in P_{2n}(\Gamma)$ *and* $S \in Hom([2n], [0])$*,*

$$S([\xi]) = \frac{\mu(v_n^\xi)}{\mu(v_{2n}^\xi)} \left(\prod_{\{i,j\} \in S : i<j\leq n} \delta_{\xi_i, \widetilde{\xi_j}} \frac{\mu(v_i^\xi)}{\mu(v_j^\xi)} \right) \left(\prod_{\{i,j\} \in S : i \leq n < j} \delta_{\xi_i, \widetilde{\xi_j}} \right) \times$$

$$\left(\prod_{\{i,j\} \in S : n<i<j} \delta_{\xi_i, \widetilde{\xi_j}} \frac{\mu(v_i^\xi)}{\mu(v_j^\xi)} \right) [(v_{2n}^\xi)]. \tag{4}$$

Sketch of Proof. As in the proof of Lemma 2, we may assume that ξ is a path consistent with S. In this case, comparison with equation (3) now shows that it suffices to see the following:

$$\prod_{\{i,j\} \in S : i<n<j} \left(\frac{\mu(v_i^\xi)}{\mu(v_j^\xi)} \right) = \frac{\mu(v_n^\xi)}{\mu(v_{2n}^\xi)}. \tag{5}$$

We illustrate by way of an example why this holds. Consider the S in Figure 1 which corresponds to the equivalence relation $\{\{1, 10\}, \{2, 7\}, \{3, 6\}, \{4, 5\}, \{8, 9\}\}$ The numbers $1, 2, \cdots, 10$ below the line index the edges of ξ while the numbers $0, 1, \cdots, 10$ above index the vertices of ξ. The LHS of equation (5) in this example is

$$\frac{\mu(v_1^\xi)}{\mu(v_{10}^\xi)} \frac{\mu(v_2^\xi)}{\mu(v_7^\xi)} \frac{\mu(v_3^\xi)}{\mu(v_6^\xi)}.$$

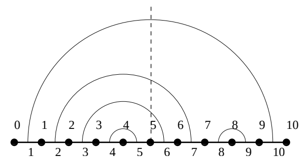

FIGURE 1. The element $S \in Hom([10],[0])$

The point now is that when the Kronecker delta terms are all non-zero, all the v_i^ξ in a single 'region' are equal. Thus in this example, $v_0^\xi = v_{10}^\xi$, $v_1^\xi = v_7^\xi = v_9^\xi$, $v_2^\xi = v_6^\xi$ and $v_3^\xi = v_5^\xi$. Hence, after cancellation, the LHS does simplify to the RHS.

Even in general, it should be clear that this happens. For the LHS of equation (5) does not depend on those classes $\{i,j\}$ of S for which both i,j are either (i) at most n or (ii) at least $n+1$. Observing that the numbers of classes satisfying (i) and (ii) are equal, we delete these classes and then we are in a situation where Lemma 2 applies. □

We will next define the algebra $F(\Gamma)$ and its structure maps. As a vector space, $F(\Gamma) = \oplus_{n\geq 0} P_n(\Gamma)$. The multiplication, denoted $\#$, is defined as follows on the path basis and extended linearly. Given $[\xi] \in P_m(\Gamma)$ and $[\eta] \in P_n(\Gamma)$, the product $[\xi]\#[\eta]$ has a component in $P_{m+n-2k}(\Gamma)$ for $0 \leq k \leq min\{m,n\}$, this component being given by

$$([\xi]\#[\eta])_{m+n-2k} = \\ \begin{cases} [\xi] \bullet [\eta] & \text{if } k = 0 \\ S_{m-k+1}^{m+n-2(k-1)} S_{m-k+2}^{m+n-2(k-2)} \cdots S_{m-1}^{m+n-2} S_m^{m+n}([\xi]\bullet[\eta]) & \text{if } k > 0 \end{cases}$$

The $*$ on $F(\Gamma)$ is exactly the same as that on $Gr(\Gamma)$ - namely $[\xi]^* = [\tilde{\xi}]$ extended conjugate linearly. Finally, define a linear functional t on $F(\Gamma)$ by setting its restriction to $P_n(\Gamma)$ for $n \geq 1$ to be 0 and by linearly extending the map $[(v)] \mapsto \mu^2(v)$ on $P_0(\Gamma)$.

Proposition 4. *$F(\Gamma)$ is a unital, associative, $*$-algebra and t is a faithful, positive trace on $F(\Gamma)$.*

Proof. A proof very similar to that in [KdySnd2008], and which we consequently omit, shows that $F(\Gamma)$ is a unital, associative $*$-algebra. To show that t is a faithful, positive trace it suffices to check that $\langle x,y\rangle = t(y^*x)$ defines an inner-product on $F(\Gamma)$ satisfying $\langle x,y\rangle = \langle y^*,x^*\rangle$. Consider the path basis $[\xi]$ of $F(\Gamma)$. It follows from the definitions and Lemma 2 that $\langle[\xi],[\eta]\rangle = \delta_{\xi,\eta}\mu(s(\xi))\mu(f(\xi))$, finishing the proof. □

We next define maps $\phi : Gr(\Gamma) \rightarrow F(\Gamma)$ and $\psi : F(\Gamma) \rightarrow Gr(\Gamma)$ as follows. Each of these restricts to maps from $P_n(\Gamma)$ to $\oplus_{m=0}^n P_m(\Gamma)$. Consequently, the maps ϕ, ψ may be represented by upper-triangular matrices $((\phi_n^m))$ and $((\psi_n^m))$ where $\phi_n^m,\ \psi_n^m : P_n(\Gamma) \rightarrow P_m(\Gamma)$ are zero if $m > n$. We define ϕ_n^m to be the (action by the) sum of all elements of $Hom([n],[m])$ and ψ_n^m to be $(-1)^{n-m}$ times the (action by the) sum of all the non-nested elements of $Hom([n],[m])$.

We now have the following proposition that identifies $Gr(\Gamma)$ and $F(\Gamma)$.

Proposition 5. *The maps ϕ and ψ define mutually inverse $*$-isomorphisms between $Gr(\Gamma)$ and $F(\Gamma)$ that intertwine the functionals τ and t.*

The proof uses the following lemma about the Kreweras complement of Temperley-Lieb equivalence relations.

Lemma 6. *Let S be a Temperley-Lieb equivalence relation on $\{1, 2, \cdots, 2n\}$ and $K(S)$ be its Kreweras complement. Then, for any class $C = \{a_1, \cdots, a_k\}$ of $K(S)$ with $a_1 < \cdots < a_k$, all the a_i have the same parity and $\{a_i + 1, a_{i+1}\} \in S$ for each $i = 1, \cdots, k$ (where $a_i + 1$ is computed modulo $2n$ and $i + 1$ is computed modulo k).*

Proof. Induce on n, with the basis case $n = 1$ following by a direct check. For $n > 1$ take $i \leq 2n - 1$ largest so that $\{i, i + 1\} \in S$. Let $T = S|_{\{1,2,\cdots,2n\}\setminus\{i,i+1\}}$. The Kreweras complement of S is obtained from that of T by adding $i + 1$ to the class of $i - 1$ and adding the singleton class $\{i\}$. Observe that $i + 1$ is the largest element in its $K(S)$ class by choice of i. Now by induction, the parity assertion holds and further, the new $\{a_i + 1, a_{i+1}\}$ that are needed to be shown to belong to S are both $\{i, i + 1\}$ which is, indeed, in S. □

Proof of Proposition 5. The proof that the maps ϕ and ψ define mutually inverse $*$-isomorphisms between $Gr(\Gamma)$ and $F(\Gamma)$ is nearly identical to that of Lemma 5.1 in [JnsShlWlk2008] and depends essentially only on properties of the category $\mathcal{E}$. We omit it here.

The intertwining assertion that needs to be checked is that $\tau = t \circ \phi$ on $Gr(\Gamma)$. Note that both sides vanish on paths of odd length and that if $[\xi]$ is a path of length $2n$, then, $\tau([\xi]) = \sum_T \tau_T([\xi])$ where the sum is over all Temperley-Lieb relations T on $\{1, 2, \cdots, 2n\}$ while $t \circ \phi([\xi]) = t(\sum_S S([\xi]))$ where the sum is over all $S \in Hom([2n],[0])$, since t vanishes on paths of positive length. The natural identification between Temperley-Lieb equivalence relations on $\{1, 2, \cdots, 2n\}$ and $Hom([2n],[0])$ shows that it suffices to see that $\tau_S([\xi]) = t \circ S([\xi])$ for any Temperley-Lieb relation S on $\{1, 2, \cdots, 2n\}$. Now both these vanish unless $s(\xi) = f(\xi)$; so we assume this. Unravelling the definitions, we need to see that under these assumptions,

$$\prod_{\{\{i,j\}\in S: i<j\}} \delta_{\xi_i,\widetilde{\xi_j}} \prod_{C\in K(S)} \mu(v_C^\xi)^{2-|C|} = \mu(v_{2n}^\xi)^2 \prod_{\{\{i,j\}\in S: i<j\}} \left(\delta_{\xi_i,\widetilde{\xi_j}} \frac{\mu(v_i^\xi)}{\mu(v_j^\xi)}\right),$$

with $K(S)$ being the Kreweras complement of S and $v_C^\xi = v_c^\xi$ for any $c \in C$. We may further assume that ξ is a path consistent with S in the sense that $\xi_i = \widetilde{\xi_j}$ whenever $\{i,j\} \in S$ and show the following

$$\mu(v_{2n}^\xi)^{-2} \prod_{C\in K(S)} \mu(v_C^\xi)^{2-|C|} = \prod_{\{\{i,j\}\in S: i<j\}} \left(\frac{\mu(v_i^\xi)}{\mu(v_j^\xi)}\right).$$

The product on the right may be rewritten as $\prod_{i=1}^{2n} \mu(v_i^\xi)^{\epsilon_S(i)}$ where $\epsilon_S(i)$ is 1 or -1 according as i is the smaller or larger element in its S-class. Next, we may regroup this product in terms of classes of $K(S)$ as $\prod_{C\in K(S)} \prod_{c\in C} \mu(v_c^\xi)^{\epsilon_S(c)}$. Now, as we have observed before, if ξ is consistent with S then all v_c^ξ for $c \in C$ are equal (to a vertex denoted v_C^ξ) and so this product now becomes $\prod_{C\in K(S)} \mu(v_C^\xi)^{\sum_{c\in C} \epsilon_S(c)}$. Comparing with the product on the left, what needs to be seen is that if C is any class of $K(S)$ then

$$\sum_{c\in C} \epsilon_S(c) = \begin{cases} 2-|C| & \text{if } 2n \notin C \\ -|C| & \text{if } 2n \in C \end{cases} \tag{6}$$

To prove equation (6), it suffices to see that for any non-external (i.e., not containing $2n$) class C of $K(S)$, $\epsilon_S(c)$ is 1 or -1 according as c is the smallest element in C or not, while for the external class, $\epsilon_S(c) = -1$ for all its elements.

But this is an easy consequence of Lemma 6. If $C = \{a_1, \cdots, a_k\}$ is a $K(S)$ class for which $a_k \neq 2n$, the definition of ϵ_S (together with Lemma 6) shows that $\epsilon_S(a_1) = 1$ while $\epsilon_S(a_i) = -1$ for $i \geq 2$. On the other hand, if $a_k = 2n$, then it similarly follows that all $\epsilon_S(a_i) = -1$, completing the proof of equation (6) and consequently of the proposition. □

An immediate consequence of Proposition 5 is the following corollary.

Corollary 7. *For any graph Γ, $(Gr(\Gamma), \tau)$ is a graded, tracial, faithful $*$-probability space.* □

We recall that if (A, τ) is a tracial probability space and $e \in A$ is a projection, then the cut-down eAe is naturally a tracial probability space where the trace is scaled so as to be 1 on e. We will find some cut-downs of $Gr(\Gamma)$ to be useful. For a vertex $v \in V$, we denote by $Gr(\Gamma, v)$ the probability space $e_v Gr(\Gamma) e_v$. Letting $e_0 = \sum_{v\in V_0} e_v$ - the sum of the projections corresponding to the even vertices - we will denote $e_0Gr(\Gamma)e_0$ by $Gr(\Gamma,0)$. Similarly, with $e_1 = \sum_{w\in V_1} e_w = 1-e_0$, we denote $e_1Gr(\Gamma)e_1$ by $Gr(\Gamma,1)$.

The bipartite nature of the graph Γ implies that the odd graded pieces of these graded algebras reduce to zero. In particular, $Gr(\Gamma,v)$ has as basis $\{[\xi] : \xi$ is a path beginning and ending at $v\}$ and is a connected graded algebra, while $Gr(\Gamma,0)$ (resp. $Gr(\Gamma,1)$) has as basis $\{[\xi] : \xi$ is a path beginning and ending at an even (resp. odd) vertex$\}$. There are also corresponding notions in the $F(\Gamma)$ picture such as $F(\Gamma,v)$, $F(\Gamma,0)$ or $F(\Gamma,1)$ and we will use self-explanatory notation such as $P_n(\Gamma,0)$ or $P_n(\Gamma,v)$. Thus, for instance, $F(\Gamma,v) = \oplus_{n\geq 0}P_{2n}(\Gamma,v)$. We will also tacitly use the fact that the isomorphism of $Gr(\Gamma)$ onto $F(\Gamma)$ of Proposition 5 takes $Gr(\Gamma,v)$ to $F(\Gamma,v)$ for each $v\in V$. In particular, it takes $Gr(\Gamma,0)$ to $F(\Gamma,0)$ and $Gr(\Gamma,1)$ to $F(\Gamma,1)$.

Consider the Hilbert space $H(\Gamma)$ obtained by completing $F(\Gamma)$ for its inner-product, which has orthonormal basis given by all

$$\{\xi\} = \frac{1}{\sqrt{\mu(s(\xi))\mu(f(\xi))}}[\xi] \tag{7}$$

where ξ is a path in Γ. Equivalently, it is the Hilbert space direct sum $\oplus_{n\geq 0}P_n(\Gamma)$ where each $P_n(\Gamma)$ has orthonormal basis $\{\xi\}$ with ξ a path in Γ of length n. We denote its norm by $||\cdot||$. We wish to show that the left regular representation of $F(\Gamma)$ on itself extends to a bounded representation on $H(\Gamma)$. It clearly suffices to see that for $a\in P_m(\Gamma)$ and $b\in F(\Gamma)$ there exists a constant C (depending only on a) such that $||a\#b|| \leq C||b||$. The proof of Proposition 4.3 in [KdySnd2008] goes over to show that even the following is sufficient (and that we may take $C=(2m+1)K$).

Proposition 8. *For $a\in P_m(\Gamma)$ and $b\in P_n(\Gamma)$ there exists a constant K (depending only on a) such that $||(a\#b)_t|| \leq K||b||$ for any t with $|m-n|\leq t\leq m+n$.*

Proof. We will work with the orthonormal basis $\{\xi\}$ rather than the orthogonal basis $[\xi]$. Observe that

$$\{\xi\}\bullet\{\eta\} = \begin{cases} 0 & \text{if } f(\xi)\neq s(\eta) \\ \frac{1}{\mu(f(\xi))}\{\xi\circ\eta\} & \text{if } f(\xi)=s(\eta). \end{cases}$$

We may assume that $a=\{\xi\}$. Suppose that $b=\sum_\eta c_\eta\{\eta\}$, where the sum is over all paths η in Γ of length n. Since $(a\#b)_t$ is obtained by an application of at most m S_i^n's to $\sum_{\{\eta:f(\xi)=s(\eta)\}}\frac{c_\eta}{\mu(f(\xi))}\{\xi\circ\eta\}$, it suffices to bound the operator norm of the S_i^n and the Hilbert space norm of $\sum_{\{\eta:f(\xi)=s(\eta)\}}\frac{c_\eta}{\mu(f(\xi))}\{\xi\circ\eta\}$.

It is easily checked that the adjoint of S_i^n is given explicitly by

$$(S_i^n)^*(\{\eta\}) = \sum_w \sum_{\rho:(v_{i-1}^\eta \xrightarrow{\rho} w)\in E} \frac{\mu(w)}{\mu(v_{i-1}^\eta)}\{\eta_{[0,i-1]}\circ\rho\circ\tilde{\rho}\circ\eta_{[i-1,n-2]}\},$$

and consequently that

$$S_i^n(S_i^n)^*(\{\eta\}) = \left(\sum_w |\{\rho : (v_{i-1}^\eta \stackrel{\rho}{\to} w) \in E\}| \left(\frac{\mu(w)}{\mu(v_{i-1}^\eta)}\right)^2\right)\{\eta\},$$

for all $\{\eta\} \in P_{n-2}(\Gamma)$. Thus, for a vertex v, if we define

$$\delta(v) = \sum_w |\{\rho : (v \stackrel{\rho}{\to} w) \in E\}| \left(\frac{\mu(w)}{\mu(v)}\right)^2,$$

and $\delta = max_{v\in V}\{\delta(v)\}$, then the operator norm of S_i^n is bounded above by $\sqrt{\delta}$.

Finally, note that $||b||^2 = \sum_\eta |c_\eta|^2$ while

$$||\sum_\eta \frac{c_\eta}{\mu(f(\xi))}\{\xi \circ \eta\}||^2 \leq \frac{1}{\mu(f(\xi))^2}\sum_\eta |c_\eta|^2 = \frac{1}{\mu(f(\xi))^2}||b||^2.$$

Thus we may take $K = \frac{\max\{1,\delta^{m/2}\}}{\mu(f(\xi))}$ for $a = \{\xi\} \in P_m(\Gamma)$ (the reason for the 'max' being to allow for the cases $\delta < 1$ and $\delta \geq 1$). $\square$

We thus have a bounded left regular representation $\lambda : F(\Gamma) \to \mathcal{L}(H(\Gamma))$ and we set $M(\Gamma) = \lambda(F(\Gamma))''$. Similarly, for $i = 0, 1$, we have the left regular representation $\lambda : F(\Gamma, i) \to \mathcal{L}(H(\Gamma, i))$ and we may define $M(\Gamma, i) = \lambda(F(\Gamma, i))''$. It is easy to see that - see Lemma 4.4 of [KdySnd2008] - each of $M(\Gamma)$, $M(\Gamma, 0)$ and $M(\Gamma, 1)$ is a finite von Neumann algebra. The goal of the next section is to show that $M(\Gamma)$ is 'almost a II_1-factor'.

2. Almost II_1-factoriality of $M(\Gamma)$

Throughout this section, our standing assumption will be that the graph Γ is connected and has at least one edge. For such a graph, it is clear that $Gr(\Gamma)$ is infinite-dimensional. The main results of this section apply only to graphs with at least two edges and show that the von Neumann algebra $M(\Gamma)$ is a direct sum of a II_1-factor and a finite-dimensional abelian algebra (possibly $\{0\}$) by analysing the local graded probability spaces $Gr(\Gamma, v)$ for each vertex $v \in V$.

In the analysis of $Gr(\Gamma, v)$, an action by a certain category that we denote by $\mathcal{C}(\delta)$ (where $\delta \in \mathbb{C}$ is some fixed non-zero parameter) will be extremely important, so we begin by describing this category.

Its objects are $[0], [1], [2], \cdots$, where we think of $[n]$ as a set of $2n$ points on a horizontal line labelled $1, 2, \cdots, 2n$. Note that the objects of $\mathcal{C}(\delta)$ are denoted by exactly the same notation as objects of $\mathcal{E}$ but mean different things.

The set $Hom([n], [m])$ is stipulated to have basis given by all $T(P, Q)_n^m$ where $P \subseteq [m]$ and $Q \subseteq [n]$ are intervals of equal cardinality, where $T([4, 5], [3, 4])_5^8 \in Hom([5], [8])$ is illustrated in Figure 2 below. The general prescription for $T(P, Q)_n^m$

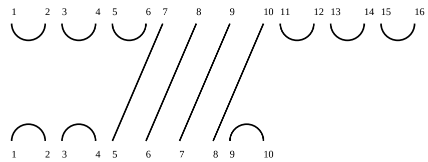

FIGURE 2. The morphism $T([4,5],[3,4])_5^8 \in Hom([5],[8])$

is the following. Points below labelled by elements of the sets $2Q$ and $2Q-1$ are joined to points above labelled by $2P$ and $2P-1$ in order preserving fashion, and the rest are capped or cupped off without nesting.

Composition in $\mathcal{C}(\delta)$ for the basis elements is as in Temperley-Lieb categories - by stacking the pictures and replacing each closed loop that appears with a multiplicative δ, thus yielding a multiple of another basis element. In order to explicitly write down the composition rule in terms of the $T(P,Q)_n^m$, note that there is an order preserving bijection $f_{PQ} : P \to Q$ such that for $p \in P$, the marked points above that are labelled by $2p-1$ and $2p$ are joined, respectively, to the points below labelled by $2f_{PQ}(p)-1$ and $2f_{PQ}(p)$. In the example above, for instance, f_{PQ} is the unique order preserving bijection from $\{4,5\}$ to $\{3,4\}$. Then the composition rule is seen to be

$$T(P,Q)_n^m \circ T(R,S)_p^n = \delta^{n-|Q\cup R|}T(Y,Z)_p^m,$$

where $Y = f_{PQ}^{-1}(Q\cap R)$ and $Z = f_{RS}(Q\cap R)$.

For $n \geq 1$ let $A_-^n \in Hom([n],[n-1])$ be the morphism with a single cap in the bottom left corner (i.e., $A_-^n = T(\{1,2,\cdots,n-1\},\{2,3,\cdots,n\})_n^{n-1}$) and $A_+^n \in Hom([n],[n-1])$ be the morphism with a single cap in the bottom right corner (i.e., $A_+^n = T(\{1,2,\cdots,n-1\},\{1,2,\cdots,n-1\})_n^{n-1}$). Similarly, for $n \geq 0$ let $C_-^n \in Hom([n],[n+1])$ be the morphism with a single cup in the top left corner and $C_+^n \in Hom([n],[n+1])$ be the morphism with a single cup in the top right corner. Thus, $C_-^n = T(\{2,3,\cdots,n+1\},\{1,2,\cdots,n\})_n^{n+1}$ and $C_+^n = T(\{1,2,\cdots,n\},\{1,2,\cdots,n\})_n^{n+1}$. (The letters A and C are meant to suggest similarity to 'annihilation' and creation'.)

Proposition 9. *If $\delta \neq 0$, the category $\mathcal{C}(\delta)$ is generated by the set*

$$\{A_\pm^n : n \geq 1\} \cup \{C_\pm^n : n \geq 0\}$$

of morphisms, and presented by the following relations

$$A^1_- = A^1_+ \tag{8}$$

$$C^0_- = C^0_+ \tag{9}$$

$$A^{n+1}_- A^{n+2}_+ = A^{n+1}_+ A^{n+2}_- \tag{10}$$

$$A^{n+1}_- C^n_- = \delta \ id_{[n]} \tag{11}$$

$$A^{n+2}_- C^{n+1}_+ = C^n_+ A^{n+1}_- \tag{12}$$

$$A^{n+2}_+ C^{n+1}_- = C^n_- A^{n+1}_+ \tag{13}$$

$$A^{n+1}_+ C^n_+ = \delta \ id_{[n]} \tag{14}$$

$$C^{n+1}_- C^n_+ = C^{n+1}_+ C^n_- \tag{15}$$

More explicitly, suppose a category $\mathcal{D}$ has the property that $Hom(D, D')$ is a complex vector space for every pair (D, D') of objects. Then, in order to specify a functor from $\mathcal{C}(\delta)$ to $\mathcal{D}$, it (is necessary and it) suffices to find objects $D_n \in \mathcal{D}$ for $n \geq 0$ and morphisms $\widetilde{A}^n_\pm : D_n \to D_{n-1}$ for $n \geq 1$ and $\widetilde{C}^n_\pm : D_n \to D_{n+1}$, for $n \geq 0$ satisfying the relations (8)-(15) above. □

Proof. It is easy to check that the relations (8)-(15) are satisfied.

To see that these are the only relations, we need to first observe that the following identities, for $k, l \geq 0$, are consequences of them:

$$A^1_- \cdots A^k_- A^{k+1}_+ \cdots A^{k+l}_+ = A^1_+ \cdots A^{k+l}_+ \tag{16}$$

$$C^{k+l-1}_+ \cdots C^k_+ C^{k-1}_- \cdots C^0_- = C^{k+l-1}_+ \cdots C^0_+ \tag{17}$$

These two identities are seen inductively to follow from the equations numbered (8) and (10), and from (9) and (15) respectively, from among the above relations.

We next describe a 'canonical form' for every morphism in $\mathcal{C}(\delta)$ as a word in the generators, in such a way that if we assign the 'rank' 1, 2, 3, and 4 to any C_+, C_-, A_+ and A_- respectively, then if the word contains generators of ranks i and j, with $i < j$, then the generator of rank i will appear to the left of the one with rank j. For example, the morphism illustrated in Figure 2) is expressed as

$$T([4,5],[3,4])^8_5 = C^7_+ C^6_+ C^5_+ C^4_- C^3_- C^2_- A^3_+ A^4_+ A^5_-$$

(The algorithm for arriving at this word is to 'start at the top right of the picture and proceed anti-clockwise till you reach the bottom right.) Notice that it is only when there are no through strings that there is an ambiguity (about whether to choose a + or a - for the A's and C's, but this is resolved using equations (16) and (17) above, using which we can demand in the case of no through string, all

C's and A's come with the subscript '+'. On the other hand, if there are through strings, the number of through strings can be read off from this word, at the point of transition from C's to A's (the number of through strings is exactly twice the superscript of the rightmost C). (By the way, it is in order to lay hands on $id_{[n]}$ that we need the condition $\delta \neq 0$.)

Finally, if the 'rank ordering' specified above is violated in any word in the generators, such instances may be rectified uniquely by using (10)-(15). (For example, any instance where an A_- (of rank 4) precedes any generator of rank 3, 2, or 1, is set right by equations numbered (10)-(12).) $\square$

We use Proposition 9 to get a functor from $\mathcal{C}(\delta(v))$ to the category $\mathcal{D}$ of $\mathbb{C}$-vector spaces and $\mathbb{C}$-linear maps. Recall that $\delta(v) = \sum_w |\{\rho : (v \xrightarrow{\rho} w) \in E\}| \left(\frac{\mu(w)}{\mu(v)}\right)^2$. Let D_n be $P_{2n}(\Gamma, v)$ and define the action of the generating morphisms as follows on $[\xi] \in P_{2n}(\Gamma, v)$.

$$\widetilde{A}^n_-([\xi]) = \delta_{\xi_1, \widetilde{\xi_2}} \frac{\mu(v^\xi_1)}{\mu(v)} \xi_{[2,2n]} \qquad \widetilde{C}^n_-([\xi]) = \sum_w \sum_{\rho:(v \xrightarrow{\rho} w) \in E} \frac{\mu(w)}{\mu(v)} [\rho \circ \tilde{\rho} \circ \xi]$$
$$\widetilde{A}^n_+([\xi]) = \delta_{\xi_{2n-1}, \widetilde{\xi_{2n}}} \frac{\mu(v^\xi_{2n-1})}{\mu(v)} \xi_{[0,2n-2]} \qquad \widetilde{C}^n_+([\xi]) = \sum_w \sum_{\rho:(v \xrightarrow{\rho} w) \in E} \frac{\mu(w)}{\mu(v)} [\xi \circ \rho \circ \tilde{\rho}]$$

A little calculation now proves the following.

Proposition 10. *The action by the generators given by the equations above extends to give a well defined functor from $\mathcal{C}(\delta(v))$ to $\mathcal{D}$.* $\square$

We now need the local Hilbert spaces $H(\Gamma, v)$ which we define to be the completions of $F(\Gamma, v)$ for their trace norms. Note that $F(\Gamma, v)$ is a (non-unital) subalgebra of $F(\Gamma)$ and that the norm on $F(\Gamma, v)$ is a scaled version of the norm on $F(\Gamma)$. The paths $[\xi]$ that begin and end at v are an orthonormal basis of $F(\Gamma, v)$ (while they have norm $\mu(v)$ regarded as elements of $F(\Gamma)$) and for this norm, $\widetilde{A}^{n+1}_-$ and $\widetilde{C}^n_-$ are adjoints of each other, as are $\widetilde{A}^{n+1}_+$ and $\widetilde{C}^n_+$.

We now try to determine the structure of the center of $Gr(\Gamma, v)$. In particular, we show that it is at most two dimensional. We will find the following notation useful. For $a \in F(\Gamma, v)$ let $[a] = \{\xi \in H(\Gamma, v) : \lambda(a)(\xi) = \rho(a)(\xi)\}$, which is a closed subspace of $H(\Gamma, v)$. Denote by Ω the vector $1 \in F(\Gamma, v) \subseteq H(\Gamma, v)$, note that this is (cyclic and) separating for $M(\Gamma, v)$ and thus the operator equation $ax = xa$ is equivalent to the vector inclusion $x\Omega \in [a]$ for $x \in M(\Gamma, v)$.

Our strategy is similar to that in [KdySnd2008], with some differences. We first define two elements $c \in P_1(\Gamma, v)$ and $d \in P_2(\Gamma, v)$. (For notational convenience we do not use the possibly more correct notation c_v and d_v.) We then show that $[c] \cap [d]$ is 1-dimensional if $\delta(v) \geq 1$ and 2-dimensional if $\delta(v) < 1$ (assuming that the graph Γ in question has at least two edges). Finally we show that in case $\delta(v) < 1$, the centre of $Gr(\Gamma, v)$ is actually 2-dimensional and that the cut-down by one of the central projections is just $\mathbb{C}$. Now some simple computations give the desired result that $M(\Gamma)$ is either a II_1-factor or a direct sum of one with a finite-dimensional abelian algebra.

In the sequel, we dispense with 'tilde's and continue to use the same symbol for morphisms in $\mathcal{C}(\delta)$ and the associated linear maps between the $P_{2n}(\Gamma, v)$.

Let $c \in P_1(\Gamma, v)$ be the element $C_-^0(1)$. Explicitly,

$$c = \sum_w \sum_{\rho:(v \xrightarrow{\rho} w) \in E} \frac{\mu(w)}{\mu(v)} [\rho \circ \tilde{\rho}].$$

Let $c_0 = 1$ and by c_n for $n \geq 1$, we will denote the element $C_-^{n-1} C_-^{n-2} \cdots C_-^0(1)$. Thus $c_n \in P_{2n}(\Gamma, v)$ and $c_1 = c$. Note that by induction on n, c_n is seen to be the highest degree term of c^n in $F(\Gamma, v)$ and to be a polynomial in c of degree n. Let $C \subseteq H(\Gamma, v)$ be the closed subspace spanned by all the $c_n \Omega$ for $n \geq 0$. We then have the following crucial result which is the analogue of Proposition 5.4 of [KdySnd2008].

Proposition 11. *$[c] = C$.*

Before sketching a proof, we state a key lemma used which is the analogue of Lemma 5.6 of [KdySnd2008]. By C_n we denote the (1-dimensional) subspace of $P_{2n}(\Gamma, v)$ spanned by c_n and by $C_n^\perp$ its orthogonal complement in $P_{2n}(\Gamma, v)$. Thus if $C^\perp$ is the orthogonal complement of C in $H(\Gamma, v)$, then $C^\perp = \oplus_{n \geq 0} C_n^\perp$.

Lemma 12. *For $n \geq 0$, the map $C_n^\perp \ni x \mapsto z = (c\#x - x\#c)_{n+1} \in P_{n+1}(\Gamma, v)$ is injective with inverse given by*

$$x = \sum_{t=1}^{n} \delta(v)^{-t} T([1, n+1-t], [t+1, n+1])^n_{n+1}(z).$$

We omit the proof except to remark that $(c\#x)_{n+1}$ and $(x\#c)_{n+1}$ are just $C_-^n(x)$ and $C_+^n(x)$. We also omit the proof of the next corollary which is the analogue of Corollary 5.7 of [KdySnd2008] with identical proof.

Corollary 13. *Suppose that $\xi = (x_0, x_1, \cdots) \in \oplus_{n=0}^\infty C_n^\perp = C^\perp$ and satisfies $\lambda(c)(\xi) = \rho(c)(\xi)$. i.e., $\xi \in C^\perp \cap [c]$. Then, for $m > n > 0$ with $m - n = 2d$, we have:*

$$\begin{aligned} x_n \quad = \quad & \sum_{t=1}^{n} \delta(v)^{-(t+d-1)} \{T([1, n+1-t], [t+d, n+d])^n_m(x_m) \\ & - T([1, n+1-t], [t+d+1, n+d+1])^n_m(x_m)\} \end{aligned}$$

One more result needed in proving Proposition 11 is the following norm estimate which is the analogue of Lemma 5.8 of [KdySnd2008].

Lemma 14. *Suppose that $x = x_n \in P_{2n}(\Gamma, v) \subseteq F(\Gamma, v)$ and let $y = T(P,Q)^m_n(x) \in P_{2m} \subseteq F(\Gamma, v)$ for some morphism $T(P,Q)^m_n$. Then $||y|| \leq \delta(v)^{\frac{1}{2}(n+m)-|A|}||x||$.*

Proof. Consider the linear extension of the map $T(P,Q)^m_n \mapsto \delta(v)^{\frac{1}{2}(n+m)-|A|}$ to $Hom([n],[m])$ for each $n, m \geq 0$. The observation is that it is multiplicative on composition. For consider $T(P,Q)^m_n \in Hom([n],[m])$ and $T(R,S)^n_p \in Hom([n],[p])$. Their composition is given by $\delta(v)^{n-|Q\cup R|}T(Y,Z)^m_p$ where $Y = f^{-1}_{PQ}(Q \cap R)$ and $Z = f_{RS}(Q \cap R)$. The multiplicativity assertion amounts to verifying that

$$\frac{1}{2}(n+m) - |P| + \frac{1}{2}(n+p) - |R| = n - |Q \cup R| + \frac{1}{2}(m+p) - |Y|,$$

which is easily verified to hold.

Hence it suffices to verify the norm estimate when $T(P,Q)^m_n$ is one of the generators $C^n_\pm$ or $A^n_\pm$ of the category $\mathcal{C}(\delta(v))$. Note that the norm estimate for all these generators is just $||y|| \leq \delta(v)^{\frac{1}{2}}||x||$. For $C^n_\pm$, we have $(C^n_\pm)^* C^n_\pm = A^{n+1}_\pm C^n_\pm = \delta(v)I^n$ while for $A^n_\pm$, we have $A^n_\pm(A^n_\pm)^* = A^n_\pm C^{n-1}_\pm = \delta(v)I^{n-1}$. This proves the norm estimate for generators and completes the proof of the lemma. □

Proof of Proposition 11. Each c_n, being a polynomial in c, clearly commutes with c and it follows that $C \subseteq [c]$. To prove the other inclusion, it is enough to see that $C^\perp \cap [c] = \{0\}$. Suppose that $\xi = (x_0, x_1, \cdots) \in C^\perp \cap [c]$. Note that $x_0 = 0$ since $x_0 \in C_0^\perp$ while $C_0 = P_0(\Gamma, 0) = \mathbb{C}$. By Corollary 13, we have for $m > n > 0$ with $m - n = 2d$,

$$\begin{aligned} x_n \quad = \quad & \sum_{t=1}^{n} \delta(v)^{-(t+d-1)} \{T([1, n+1-t], [t+d, n+d])^n_m(x_m) \\ & - T([1, n+1-t], [t+d+1, n+d+1])^n_m(x_m)\} \end{aligned}$$

Now, applying the norm estimate from Lemma 14 and using the triangle inequality gives $||x_n|| \leq 2n||x_m||$. Now $\xi \in H(\Gamma, v) \Rightarrow lim_{m\to\infty}||x_m|| = 0$ and so $x_n = 0$ for all $n > 0$. Hence $\xi = 0$, completing the proof. □

We now consider the element $d \in P_2(\Gamma, v)$ defined explicitly by

$$d = \sum_w \sum_{\rho:(v \xrightarrow{\rho} w)\in E} \sum_x \sum_{\zeta:(w \xrightarrow{\zeta} x)\in E} \frac{\mu(x)}{\mu(v)} [v \xrightarrow{\rho} w \xrightarrow{\zeta} x \xrightarrow{\tilde{\zeta}} w \xrightarrow{\tilde{\rho}} v].$$

Loosely, d can be thought of as pictorially represented as in Figure 3.

It must be noted that the action of the category $\mathcal{C}(\delta)$ on $F(\Gamma, v)$ may be extended to an action of the entire Temperley-Lieb category on $F(\Gamma, v)$ if μ^2 is a Perron-Frobenius eigenvector for the incidence matrix of Γ, however, this may no longer

FIGURE 3. Pictorial representation of d

be possible if the 'Perron-Frobenius assumption' is dropped. Since we shall have to address that situation, we do not assume this. Nevertheless, we will see that in situations of interest for us, this pictorial representation will be of heuristic value.

We wish to consider a special element of $M(\Gamma, v)$ defined when $\delta(v) < 1$. Since the proof of existence of this element requires a careful norm estimate, we digress with the necessary lemma.

Lemma 15. *Suppose the weighting on* Γ *is such that* $\delta(v) < 1$. *The sequence of elements* $x_m = \sum_{n=0}^{m}(-1)^n c_n \in F(\Gamma, v) \subseteq$[3] $M(\Gamma, v) \subseteq \mathcal{L}(H(\Gamma, v))$ *converge in the strong operator topology. Hence the series* $\sum_{n=0}^{\infty}(-1)^n c_n$ *defines an element* $z_v \in M(\Gamma, v)$.

Proof. It suffices to see that the x_m are uniformly bounded in norm and that for ξ in a dense subspace of $H(\Gamma, v)$, the sequence $x_m\xi$ converges in $H(\Gamma, v)$. Note that if $\xi = \Omega$ - the vacuum vector of $H(\Gamma, v)$ - then, $x_m\xi = \sum_{n=0}^{m}(-1)^n c_n\Omega$. Since $||c_n\Omega||^2 = \delta(v)^n$, when $\delta(v) < 1$, the $x_m\xi$ converge in $H(\Gamma, v)$. It follows that on the dense subspace $M(\Gamma, v)'\Omega$ too, we obtain convergence. It remains to prove the norm estimate.

Consider the block matrix representing multiplication by c_n on the Hilbert space $H(\Gamma, v)$ with respect to the orthogonal decomposition $H(\Gamma, v) = \oplus_{n=0}^{\infty} P_{2n}(\Gamma, v)$. The definition of multiplication in $F(\Gamma, v)$ shows that for any path ξ of length $2m$,

$$c_n\#[\xi] = \sum_{k=0}^{min\{2m,2n\}} C^{n-\lceil \frac{k}{2}\rceil} A^{\lfloor \frac{k}{2}\rfloor}[\xi]$$

where, to avoid heavy notation, for $[\xi] \in P_{2n}(\Gamma, v)$ we write $C^pA^q[\xi]$ to mean

$$C_-^{n-q+p-1} \cdots C_-^{n-q+1} C_-^{n-q} A_-^{n-q+1} \cdots A_-^{n-1} A_-^{n}[\xi].$$

Thus the (i, j)-block (note that $i, j \geq 0$) of the matrix for c_n is 0 unless $|i - j| \leq n \leq i + j$, in which case it is given by

$$C^{\lfloor \frac{n-j+i}{2} \rfloor} A^{\lfloor \frac{n+j-i}{2} \rfloor}.$$

It now follows that the (i, j)-block of the matrix for x_m is given by

$$\sum_{|i-j| \leq n \leq min\{m, i+j\}} (-1)^n C^{\lfloor \frac{n-j+i}{2} \rfloor} A^{\lfloor \frac{n+j-i}{2} \rfloor}$$

[3] This inclusion is, naturally, via λ.

Every odd term of this sum (starting with the first which corresponds to $n = |i - j|$) is equal, except for sign, with the succeeding even term and so the sum vanishes when there are an even number of terms and equals its last term when there are an odd number of terms. Note that the number of terms is $min\{m, i+j\} - |i-j| + 1$.

We consider two cases depending on whether $m \leq i+j$ or $m > i+j$.

Case I: If $m > i+j$, the number of terms is certainly odd and so

$$(x_m)^i_j = (-1)^{i+j} C^i A^j$$

Case II: If $i + j \geq m$ the number of terms is odd or even according as m and $i + j$ have the same parity or not and so, in this case,

$$(x_m)^i_j = \begin{cases} 0 & \text{if } m < |i-j| \text{ or } (i+j) - m \text{ is odd} \\ (-1)^m C^{\lfloor \frac{m-j+i}{2} \rfloor} A^{\lfloor \frac{m+j-i}{2} \rfloor} & \text{otherwise} \end{cases}$$

For instance, the matrix for x_2 is given by

$$\begin{pmatrix} I & -A & A^2 & 0 & 0 & \cdots \\ -C & CA & 0 & A^2 & 0 & \cdots \\ C^2 & 0 & CA & 0 & A^2 & \cdots \\ 0 & C^2 & 0 & CA & 0 & \cdots \\ 0 & 0 & C^2 & 0 & CA & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Observe that the (i, j) entry of x_m is non-zero only if $|i - j| \leq m$ in which case it is of the form $\pm C^p A^q$ with $p+q \geq |i-j|$. Since each of A and C has norm bounded above by $\delta(v)^{\frac{1}{2}}$, the diagonal of x_m with $i - j = t$ has norm at most $\delta(v)^{\frac{|t|}{2}}$ and so x_m itself has norm bounded by $\sum_{t=-m}^{m} \delta(v)^{\frac{|t|}{2}} \leq 1 + 2\sum_{t=1}^{\infty} \delta(v)^{\frac{t}{2}}$. Thus the x_m are uniformly bounded in norm, finishing the proof. □

For reasons that will become clear in Proposition 17, we define (in the notation introduced in Lemma 15),

$$e^I_v = \begin{cases} 0 & \text{if } \delta(v) \geq 1 \\ (1 - \delta(v)) z_v & \text{if } \delta(v) < 1 \end{cases}$$

and $e^{II}_v = e_v - e^I_v$.

The next proposition is the analogue of Proposition 5.5 of [KdySnd2008]; the reader is urged to compare the proof of that proposition with this one. For the rest of this section, Γ will always denote a connected graph with at least two edges.

Proposition 16. *Suppose that Γ is a connected graph with at least two edges. Then, $[c] \cap [d]$ has basis $\{1\}$ if $\delta(v) \geq 1$ and basis $\{1, z_v\}$ if $\delta(v) < 1$.*

Proof. Suppose that $\xi = (x_0, x_1, \cdots) \in [c] \cap [d]$. Since $\xi \in [c] = C$ by Proposition 11, there exist scalars $y(n) \in \mathbb{C}$ such that $x_n = y(n)c_n$ and since $\xi \in H(\Gamma, v)$, we have that $||\xi||^2 = \sum_{n=0}^{\infty} |y(n)|^2 \delta(v)^n < \infty$.

Some computation now shows that for $t > 2$, the $P_{2t}(\Gamma, v)$ component of $\lambda(d)([\xi]) - \rho(d)([\xi])$ is given by $(y(t-1)+y(t-2))(d\#c_{t-2} - c_{t-2}\#d)_t$. Writing out $(d\#c_{t-2} - c_{t-2}\#d)_t$ for $t > 2$ in terms of the path basis, inspection shows that it cannot vanish since Γ has at least two edges. Thus each $y(t-1) + y(t-2)$ vanishes for $t > 2$.

Hence if $y(1) = y$, then all odd $y(n) = y$ and all even $y(n) = -y$ (for $n > 0$). Now the norm condition that $\delta(v) \geq 1 \Rightarrow y = 0$ so that ξ is a multiple of $(1, 0, 0, \cdots)$ while if $\delta(v) < 1$, then ξ is a linear combination of 1 and z_v. □

We now justify the choice of notation for e_v^I and e_v^{II}.

Proposition 17. *Suppose that Γ is a connected graph with at least two edges. Then,*

1. *If $\delta(v) < 1$, then $Z(M(\Gamma, v))$ is 2-dimensional and has basis $\{e_v^I, e_v^{II}\}$. The element $e_v^I \in M(\Gamma, v)$ is a minimal (and central) projection.*
2. *e_v^{II} is a minimal central projection in $M(\Gamma, v)$ and $e_v^{II} M(\Gamma, v) e_v^{II}$ is a II_1-factor; in particular, $M(\Gamma, v)$ is a factor if $\delta(v) \geq 1$.*

Proof. (1) Since $Z(M(\Gamma, v))\Omega \subseteq [c] \cap [d]$ and $Z(M(\Gamma, v)) \to Z(M(\Gamma, v))\Omega$ is injective, Proposition 16 implies that $Z(M(\Gamma, v))$ is at most 2-dimensional.

If $\delta(v) < 1$, it follows from Proposition 16 that $Z(M(\Gamma, v))$ is at most two dimensional and contained in the span of e_v (the identity for $M(\Gamma, v)$) and e_v^I.

We shall regard c_n as an element of $e_v F(\Gamma) e_v \subset F(\Gamma)$ (so that $c_0 = e_v$) and consider a length one path $\xi = [w \stackrel{\alpha}{\to} x]$ of $F(\Gamma)$. Calculation shows that

$$c_n\#[\xi] = c_n \bullet [\xi] + c_{n-1} \bullet [\xi]$$

and therefore that

$$z_v\#[\xi] = 0$$

Similarly, (or by simply taking adjoints) we find that also

$$[\xi]\#z_v = 0$$

Associativity of multiplication implies that these equations hold even when $[\xi]$ is a path of length greater than 1 from w to x.

Finally, if ξ is a path of length 0, then $\xi = e_w$ for some w, and

$$z_v\#[\xi] = [\xi]\#z_v = \begin{cases} 0 & \text{if } w \neq v \\ z_v & \text{if } w = v \end{cases} .$$

It follows easily that $z_v \in Z(M(\Gamma))$ and that

$$z_v M(\Gamma) = \mathbb{C} z_v \tag{18}$$

Taking adjoints yields

$$M(\Gamma)z_v = \mathbb{C}z_v \tag{19}$$

Deduce that there is some constant $\gamma > 0$ such that $z_v^2 = \gamma z_v$.

By comparing their inner products (in $\mathcal{H}(\Gamma, v)$) with e_v, we find that

$$\sum_{n=0}^{\infty} (\delta(v))^n = \gamma$$

and hence that $\gamma = (1 - \delta(v))^{-1}$. Thus we find that indeed e_v^I is a projection in $M(\Gamma)$ which is central and minimal, since

$$e_v^I M(\Gamma) e_v^I = \mathbb{C}e_v^I. \tag{20}$$

(2) It is seen from Proposition 16 that in both cases, e_v^{II} is a minimal central projection. The assumed non-triviality of Γ ensures that $M(\Gamma, v)$ is an infinite-dimensional but finite von Neumann algebra, it is seen (from the minimality of e_v^I in case $\delta(v) < 1$) that the localisation $e^{II} M(\Gamma, v) e^{II}$ is a II_1 factor. □

Corollary 18. *For distinct vertices v and w, we have*

$$e_v^I M(\Gamma) e_w = \{0\} = e_w M(\Gamma) e_v^I$$

Proof. We only need to prove the nontrivial case when $\delta(v) < 1$. First deduce from equation (18) (when $\delta(v) < 1$) and the definition of e_v^I (when $\delta(v) \geq 1$) that $e_v^I M(\Gamma) = \mathbb{C}e_v^I$. Hence, $e_v^I M(\Gamma) e_w = \mathbb{C}e_v^I e_w = 0$. The second assertion of the corollary is obtained by taking adjoints in the first. □

Before proceeding to the next corollary to Proposition 17 we digress with an elementary fact about local and global behaviour of von Neumann algebras.

Lemma 19. *Suppose $\{p_i : i \in I\}$ is a partition of the identity element into a family of pairwise orthogonal projections of a von Neumann algebra M.*

Then the following conditions are equivalent:

1. *M is a factor;*
2. *$p_i M p_i$ is a factor for all $i \in I$, and $p_i M p_j \neq \{0\}$ $\forall i, j$.*

Proof. We only indicate the proof of the non-trivial implication (2) $\Rightarrow$ (1). For this, suppose $x \in Z(M)$. Let us write $x_{ij} = p_i x p_j$. The assumption (2) clearly implies that (i) $x_{ij} = 0$ for $i \neq j$ (since x commutes with each p_i); and (ii) for each $i \in I$, $x_{ii} = \lambda_i p_i$ for some $\lambda_i \in \mathbb{C}$. Fix an arbitrary pair (i, j) of distinct indices from I. By assumption, there exists a non-zero $y \in M$ satisfying $y = p_i y p_j$. The requirement $xy = yx$ is seen to now imply that $\lambda_i y = \lambda_j y$ and hence that $\lambda_i = \lambda_j$, and this is true for all i, j. Hence, $x = \sum_i \lambda_i p_i \in \mathbb{C}1$. □

Corollary 20. *Assume that Γ is a connected graph with at least two edges. Then,*

(1) $e_v^{II} M(\Gamma) e_w^{II} \neq 0$ *for all vertices* v, w
(2) *If we let* $e^I = \sum_{v \in V} e_v^I$*, then* $e^I M(\Gamma) e^I = \bigoplus_{v \in V} \mathbb{C} e_v^I$.
(3) *If we let* $e^{II} = \sum_{v \in V} e_v^{II}$*, then* $e^{II} M(\Gamma) e^{II}$ *is a* II_1 *factor.*

Proof. (1) Since Γ is connected, we can find a finite path ξ with $s(\xi) = v$ and $f(\xi) = w$. Then, deduce from Corollary 18 that

$$\begin{aligned} 0 &\neq [\xi] \\ &= e_v \# [\xi] \# e_w \\ &= e_v^{II} \# [\xi] \# e_w^{II} \end{aligned}$$

(2) First observe that as $e_w^I \leq e_w$ for all $w \in V$, it follows that

$$\begin{aligned} v \neq w \quad \Rightarrow \quad & e_v^I M(\Gamma) e_w^I \\ = \quad & e_v^I M(\Gamma) e_w e_w^I \\ = \quad & 0 \end{aligned}$$

by Corollary 18.

On the other hand

$$e_v^I M(\Gamma) e_v^I = \mathbb{C} e_v^I$$

The desired assertion follows from the orthogonality of the e_v^I's.

(3) As has already been observed, $e_v M(\Gamma) e_v = M(\Gamma, v)$ since $M(\Gamma)$ (resp., $M(\Gamma, v)$) is generated (as a von Neumann algebra) by the set of all finite paths $[\xi]$ (resp., those paths which start and finish at v, i.e., which satisfy $[\xi] = e_v \# [\xi] \# e_v$). Given this observation, the assertion to be proved is seen to follow from Proposition 17, Lemma 19 and the already established part (1) of this Corollary. □

We have finally arrived at the main result of this section, whose statement uses the foregoing notation and which is an immediate consequence of Corollary 20.

Theorem 21. *Assume Γ is a connected graph with at least two edges. Then, we have the following isomorphism of non-commutative probability spaces:*

$$M(\Gamma) \cong M \oplus \bigoplus_{\{v \in V : \delta(v) < 1\}} \overset{\mathbb{C} e_v^I}{(1-\delta(v))\mu^2(v)}$$

where M is some II_1 factor (and we have omitted mention of the obvious value of the trace-vector on the M-summand for typographical reasons).

Proof. It follows from Proposition 17(1), Corollary 18 and the fact that $1 = \sum_v e_v = \sum_v e_v^I + e^{II}$ that each $e_v^I \in Z(M)$ and that consequently both e^I and e^{II} are central. The asserted conclusion follows now from Corollary 20. □

Corollary 22. *If Γ is as in Theorem 21, and is equipped with the 'Perron-Frobenius weighting', then $M(\Gamma)$ is a II_1 factor.*

Proof. The hypotheses ensure that for all v, $\delta(v) = \delta$ is the Perron-Frobenius eigenvalue of the adjacency matrix of Γ, which in turn is greater than one, so the second summand of Theorem 21 is absent. □

3. Structure of the even graded probability space

In this section we let Γ be any finite, weighted, bipartite graph and regard $Gr(\Gamma, 0)$ as an operator valued probability space over its subalgebra $P_0(\Gamma, 0)$ - the abelian algebra with minimal central projections given by all e_v where $v \in V_0$ is an even vertex. Our goal is to express this as a(n algebraic) free product with amalgamation over $P_0(\Gamma, 0)$ of simpler subalgebras.

We briefly summarise from [Spc1998] the theory of operator valued probability spaces and operator valued free cumulants. An operator valued probability space is a unital inclusion of unital algebras $B \subseteq A$ equipped with a $B-B$-bimodule map $\phi : A \to B$ with $\phi(1) = 1$. A typical example is $N \subseteq M$ where M is a von Neumann algebra with a faithful, normal, tracial state τ and ϕ is the τ-preserving conditional expectation.

The lattice of non-crossing partitions plays a fundamental role in the definition of free cumulants. Recall that for a totally ordered finite set Σ, a partition π of Σ is said to be non-crossing if whenever $i < j$ belong to a class of π and $k < l$ belong to a different class of π, then it is not the case that $k < i < l < j$ or $i < k < j < l$. The collection of non-crossing partitions of Σ, denoted $NC(\Sigma)$, forms a lattice for the partial order defined by $\pi \geq \rho$ if π is coarser than ρ or equivalently, if ρ refines π. The largest element of the lattice $NC(\Sigma)$ is denoted 1_Σ. Explicitly, $1_\Sigma = \{\Sigma\}$. If $\Sigma = [n] \stackrel{\text{def}}{=} \{1, 2, \cdots, n\}$ for some $n \in \mathbb{N}$, we will write $NC(n)$ and 1_n for $NC(\Sigma)$ and 1_Σ respectively.

Before defining operator valued free cumulants, we state a basic combinatorial result that we will refer to as Möbius inversion. Suppose that A is an operator valued probability space over B. Let $X \subseteq A$ be a $B-B$-submodule and suppose given $B-B$-bimodule maps $\phi_n : \otimes_B^n X \to B$. By the multiplicative extension of this collection, we will mean the collection of $B-B$-bimodule maps $\{\phi_\pi : \otimes_B^n X \to B\}_{n \in \mathbb{N}, \pi \in NC(n)}$ defined recursively by

$$\phi_\pi(x^1 \otimes x^2 \otimes \cdots \otimes x^n) = \begin{cases} \phi_n(x^1 \otimes x^2 \otimes \cdots \otimes x^n) \\ \phi_\rho(x^1 \otimes \cdots \otimes x^{k-1} \otimes x^k \phi_{l-k}(x^{k+1} \otimes \cdots \otimes x^l) \otimes x^{l+1} \otimes \cdots \otimes x^n), \end{cases}$$

according as $\pi = 1_n$ or $\pi = \rho \cup 1_{[k+1,l]}$ for $\rho \in NC([1,n] \setminus [k+1,l])$. A little thought shows that the multiplicative extension is well-defined. Let $\mu(\cdot,\cdot)$ be the Möbius function of the lattice $NC(n)$ - see Lecture 10 of [NcaSpc2006].

Proposition 23. *Given two collections of $B-B$-bimodule maps $\{\phi_n : \otimes_B^n X \to B\}_{n\in\mathbb{N}}$ and $\{\kappa_n : \otimes_B^n X \to B\}_{n\in\mathbb{N}}$ extended multiplicatively, the following conditions are all equivalent:*

(1) $\phi_n = \sum_{\pi \in NC(n)} \kappa_\pi$ *for each* $n \in \mathbb{N}$.
(2) $\kappa_n = \sum_{\pi \in NC(n)} \mu(\pi, 1_n)\phi_\pi$ *for each* $n \in \mathbb{N}$.
(3) $\phi_\tau = \sum_{\pi \in NC(n), \pi \le \tau} \kappa_\pi$ *for each* $n \in \mathbb{N}, \tau \in NC(n)$.
(4) $\kappa_\tau = \sum_{\pi \in NC(n), \pi \le \tau} \mu(\pi, \tau)\phi_\pi$ *for each* $n \in \mathbb{N}, \tau \in NC(n)$.

Sketch of Proof. Clearly (3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (2) by taking $\tau = 1_n$. Next, suppose (2) is given. We will prove (4) by induction on the number of classes of τ. The basis case when $\tau = 1_n$ is clearly true. If, on the other hand, $\tau = \rho \cup 1_{[k+1,l]}$ for $\rho \in NC([1,n] \setminus [k+1,l])$, we compute

$$\begin{aligned} &\kappa_\tau(x^1 \otimes x^2 \otimes \cdots \otimes x^n) \\ &= \kappa_\rho(x^1 \otimes \cdots \otimes x^{k-1} \otimes x^k \kappa_{l-k}(x^{k+1} \otimes \cdots \otimes x^l) \otimes x^{l+1} \otimes \cdots \otimes x^n) \\ &= \sum_{\substack{\lambda \in NC([1,n]\setminus[k+1,l]), \lambda \le \rho \\ \nu \in NC([k+1,l])}} \mu(\lambda, \rho)\mu(\nu, 1_{l-k}) \times \\ &\qquad \phi_\lambda(x^1 \otimes \cdots \otimes x^{k-1} \otimes x^k \phi_\nu(x^{k+1} \otimes \cdots \otimes x^l) \otimes x^{l+1} \otimes \cdots \otimes x^n) \\ &= \sum_{\pi \in NC(n), \pi \le \tau} \mu(\pi, \tau)\phi_\pi(x^1 \otimes x^2 \otimes \cdots \otimes x^n). \end{aligned}$$

Here, the first equality is by the multiplicativity of κ; the second follows by (two applications of) the inductive assumption; and the third equality follows from (i) the identification $[0_n, \tau] = [0_{[1,n]\setminus[k+1,l]}, \rho] \times [0_{[k+1,l]}, 1_{[k+1,l]}]$ of posets, (ii) the fact that μ is 'multiplicative' with respect to such decompositions of 'intervals'; and from (iii) the multiplicativity of ϕ. This finishes the inductive step and hence proves (4). An even easier proof shows that (1) $\Leftrightarrow$ (3). Finally, (3) $\Leftrightarrow$ (4) by usual Mobius inversion in the poset $NC(n)$. □

Definition 24. The free cumulants of a B-valued probability space (A, ϕ) are the $B-B$-bimodule maps $\kappa_n : \otimes_B^n A \to B$ associated as in Proposition 23 to the collection of $B-B$-bimodule maps $\{\phi_n : \otimes_B^n A \to B\}_{n\in\mathbb{N}}$ defined by $\phi_n(a^1 \otimes \cdots \otimes a^n) = \phi(a^1a^2\cdots a^n)$.

The importance of the operator valued free cumulants lies in the following theorem of Speicher linking their vanishing to freeness with amalgamation over the base.

Theorem 25. *Let (A, ϕ) be a B-valued probability space and $\{A_i : i \in I\}$ be a family of B-subspaces of A such that A_i is generated as an algebra over B by $G_i \subseteq A_i$. This family is freely independent with amalgamation over B iff for each positive integer k, indices $i_1, \cdots, i_k \in I$ that are not all equal and elements $a_t \in G_{i_t}$ for $t = 1, 2, \cdots, k$, the equality $\kappa_k(a_1 \otimes a_2 \otimes \cdots \otimes a_k) = 0$ holds.* □

We will regard $P_0(\Gamma, 0) \subseteq Gr(\Gamma, 0)$ as an operator valued probability space with the map ϕ being defined on $P_n(\Gamma, 0)$ by the sum of the action of all $Hom([n], [0])$ morphisms. Equivalently, it is the transport to the $Gr(\Gamma, 0)$ picture of the map given in the $F(\Gamma, 0)$ picture by the 'orthogonal projection to $P_0(\Gamma, 0)$'. This is easily checked to be an identity preserving $P_0(\Gamma, 0) - P_0(\Gamma, 0)$-bimodule map. Further, it preserves the faithful, positive trace τ, as is checked by definiton of t in the $F(\Gamma, 0)$ picture, and transporting to $Gr(\Gamma, 0)$.

In order to state the main result of this section, we need to introduce some notation. Observe first that $Gr(\Gamma, 0)$ is generated as an algebra by $P_0(\Gamma, 0)$ and all $[\xi]$ where ξ is a path length 2 in Γ. For any odd vertex $w \in V_1$ (resp. even vertex $v \in V_0$), let Γ_w (resp. Γ_v) denote the subgraph of Γ induced on the vertex set $V_0 \cup \{w\}$ (resp., $\{v\} \cup V_1$). The μ function of Γ_w (resp. Γ_v) is the (appropriately normalised) restricted μ function of Γ. Then, $Gr(\Gamma_w, 0)$ is naturally isomorphic - as a $*$-probability space - to the subalgebra of $Gr(\Gamma, 0)$ generated by $P_0(\Gamma, 0)$ and all $[\xi]$ such that $l(\xi) = 2$ and $v_1^\xi = w$, i.e., paths of length 2 with middle vertex w. We will refer to this subalgebra as $Gr(\Gamma_w, 0)$.

Our main result in this section is then the following proposition.

Proposition 26. *For an arbitrary finite, weighted, bipartite graph Γ, we have*

$$Gr(\Gamma, 0) = *_{P_0(\Gamma,0)}\{Gr(\Gamma_w, 0) : w \in V_1\}.$$

The crucial step in the proof of this proposition is the identification of the $P_0(\Gamma, 0)$-valued free cumulants on the generators $[\xi]$ of $Gr(\Gamma, 0)$, which is done in the next proposition. In this, we will use a natural bijection $\pi \mapsto S(\pi)$ between the sets $NC(\Sigma)$ and $TL(\Sigma \times \{1, 2\})$ (for a totally ordered finite set Σ and where we consider the dictionary order on $\Sigma \times \{1, 2\}$) defined as follows.

Suppose that $\pi \in NC(\Sigma)$. Let C be a class of π and enumerate the elements of C in increasing order as, say, $C = \{c_1, c_2, \cdots, c_t\}$. Decree $\{(c_1,2),(c_2,1)\}$, $\{(c_2,2),(c_3,1)\}$, $\cdots$, $\{(c_{t-1},2),(c_t,1)\}$, $\{(c_t,2),(c_1,1)\}$ to be classes of $S(\pi)$. Do this for each class of π to define $S(\pi)$. Observe that $S(\pi)$ is a union of equivalence relations on $C \times \{0,1\}$ as C varies over classes of S. If $\Sigma = \{1,2,\cdots,n\}$, we will regard $S(\pi)$ as an element of $TL(\{1,2,\cdots,2n\})$ or equivalently as an an element of $Hom([2n],[0])$, via the obvious order isomorphisms. We illustrate with an example. Suppose that $\pi = \{\{1,6\},\{2,3,4,5\}\}$. Then, $S(\pi)$ is shown

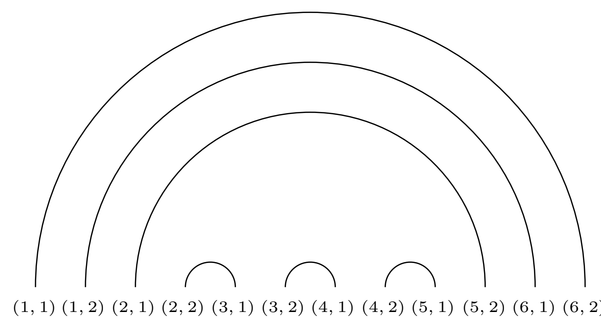

FIGURE 4. The Temperley-Lieb equivalence relation $S(\pi)$

in Figure 4. Regarded as a Temperley-Lieb relation on $\{1,2,\cdots,12\}$, $S(\pi) = \{\{1,12\},\{2,11\},\{3,10\},\{4,5\},\{6,7\},\{8,9\}\}$.

We will also use the notion of a starry path in Γ by which we mean an even length path $\xi = (v_0^\xi \stackrel{\xi_1}{\to} v_1^\xi \stackrel{\xi_2}{\to} v_2^\xi \stackrel{\xi_3}{\to} \cdots \stackrel{\xi_{2n}}{\to} v_{2n}^\xi)$, where $\xi_{2i} = \widetilde{\xi_{2i+1}}$ (indices modulo $2n$) for $i = 1,2,\cdots,n$. In such a path, we have $v_0^\xi = v_{2n}^\xi$ and all the odd v_i^ξ are equal (to the centre of the 'star'). Note that a path ξ of length $2n$ is starry exactly when $S(1_n)([\xi]) \neq 0$ (in which case, it is a scalar multiple of e_v where v is the start and finish point of ξ.)

Proposition 27. *Let $\xi^1, \xi^2, \cdots, \xi^n$ be paths in Γ of length 2. The $P_0(\Gamma,0)$-valued free cumulant on $Gr(\Gamma,0)$ is given thus: $\kappa_n([\xi^1],[\xi^2],\cdots,[\xi^n])$ is non-zero only when $\xi = \xi^1 \circ \xi^2 \circ \cdots \circ \xi^n$ is a starry path in Γ (in particular this composition should make sense), in which case, all the odd v_i^ξ are equal to some $w \in V_1$ and $v_0^\xi = v_{2n}^\xi$ is some $v \in V_0$, and*

$$\begin{aligned}\kappa_n([\xi^1],[\xi^2],\cdots,[\xi^n]) &= S(1_n)([\xi]) \\ &= \frac{\mu(v_2^\xi)\mu(v_4^\xi)\cdots\mu(v_{2n-2}^\xi)}{\mu(w)^{n-2}\mu(v)} e_v.\end{aligned}$$

Proof. Consider the $P_0(\Gamma,0)-P_0(\Gamma,0)$-bimodule $P_2(\Gamma,0)$ and define for each $n \geq 1$, $\widetilde{\kappa}_n : P_2(\Gamma,0) \times P_2(\Gamma,0) \times \cdots \times P_2(\Gamma,0) \to P_0(\Gamma,0)$ by the $\mathbb{C}$-multilinear extension of the prescription given on basis elements by the statement of the proposition. It is easily checked that $\tilde{\kappa}_n$ induces a bimodule map also denoted $\tilde{\kappa}_n : \otimes^n_{P_0(\Gamma,0)} P_2(\Gamma, 0) \to P_0(\Gamma, 0)$.

We will check that $\widetilde{\kappa}_n$ agrees on $\otimes^n_{P_0(\Gamma,0)} P_2(\Gamma, 0)$ with the operator valued cumulant κ_n. In view of Proposition 23, it suffices to check that if $\widetilde{\kappa}_\pi$ is the multiplicative extension of $\widetilde{\kappa}$, then,

$$\phi([\xi^1] \bullet [\xi^2] \bullet \cdots \bullet [\xi^n]) = \sum_{\pi \in NC(n)} \widetilde{\kappa}_\pi([\xi^1] \otimes \cdots \otimes [\xi^n])$$

for all paths $\xi^1, \cdots, \xi^n$ of length 2 in Γ.

Observe first that both sides of the equation above vanish unless $\xi^1 \circ \xi^2 \circ \cdots \circ \xi^n$ makes sense and defines a path ξ with equal end-points, so we suppose this to be the case. By definition then, the left hand side $\phi([\xi^1] \bullet [\xi^2] \bullet \cdots \bullet [\xi^n])$ of the desired equality is given by $\sum_{S \in Hom([2n],[0])} S([\xi^1 \circ \xi^2 \circ \cdots \circ \xi^n])$. In view of the natural bijection between $NC(n)$ and $Hom([2n], [0])$ alluded to above, it clearly suffices to see that

$$\widetilde{\kappa}_\pi([\xi^1] \otimes \cdots \otimes [\xi^n]) = S(\pi)([\xi^1 \circ \xi^2 \circ \cdots \circ \xi^n]). \tag{21}$$

Observe that for $\pi \in NC(n)$ and paths $\xi^1, \xi^2, \cdots, \xi^n$ of length 2 for which the composite $\xi^1 \circ \cdots \circ \xi^n$ is defined,

$$\begin{aligned} &S(\pi)([\xi^1 \circ \xi^2 \circ \cdots \circ \xi^n]) = \\ &\left(\prod_{C=\{c_1,\cdots,c_t\}\in\pi} \left\{ \delta_{\xi_2^{c_t}, \widetilde{\xi_1^{c_1}}} \frac{\mu(v_1^{\xi^{c_1}})}{\mu(v_2^{\xi^{c_t}})} \prod_{p=1}^{t-1} \left(\delta_{\xi_2^{c_p}, \widetilde{\xi_1^{c_{p+1}}}} \frac{\mu(v_2^{\xi^{c_p}})}{\mu(v_1^{\xi^{c_{p+1}}})} \right) \right\} \right) e_{v_0^{\xi^1}} \end{aligned} \tag{22}$$

(where $c_1 < c_2 < \cdots < c_t$). For instance, for the S shown in Figure 4, we have

$$\begin{aligned} S(\pi)([\xi^1 \circ \xi^2 \circ \cdots \circ \xi^n]) \quad = \quad & \left\{ \delta_{\xi_2^6, \widetilde{\xi_1^1}} \frac{\mu(v_1^{\xi^1})}{\mu(v_2^{\xi^6})} \delta_{\xi_2^1, \widetilde{\xi_1^6}} \frac{\mu(v_2^{\xi^1})}{\mu(v_1^{\xi^6})} \right\} \times \\ & \left\{ \delta_{\xi_2^5, \widetilde{\xi_1^2}} \frac{\mu(v_1^{\xi^2})}{\mu(v_2^{\xi^5})} \delta_{\xi_2^2, \widetilde{\xi_1^3}} \frac{\mu(v_2^{\xi^2})}{\mu(v_1^{\xi^3})} \delta_{\xi_2^3, \widetilde{\xi_1^4}} \frac{\mu(v_2^{\xi^3})}{\mu(v_1^{\xi^4})} \delta_{\xi_2^4, \widetilde{\xi_1^5}} \frac{\mu(v_2^{\xi^4})}{\mu(v_1^{\xi^5})} \right\} e_{v_0^{\xi^1}}. \end{aligned}$$

We prove equation (21) by induction on the number of classes of π. In case $\pi = 1_n$, it holds by definitions of $\widetilde{\kappa}_\pi$ and S. Suppose next that $\pi = \rho \cup 1_{[k+1,l]}$ for $\rho \in NC([1, n] \setminus [k + 1, l])$. Since $\widetilde{\kappa}$ multiplicatively extends $\{\widetilde{\kappa_n}\}_{n\in\mathbb{N}}$ we have,

$$\widetilde{\kappa}_\pi([\xi^1], \cdots, [\xi^n]) = \widetilde{\kappa}_\rho([\xi^1], \cdots, [\xi^{k-1}], [\xi^k]\widetilde{\kappa}_{l-k}([\xi^{k+1}], \cdots, [\xi^l]), [\xi^{l+1}], \cdots, [\xi^n]).$$

By definition,

$$\widetilde{\kappa}_{l-k}([\xi^{k+1}], \cdots, [\xi^l]) = \left\{ \delta_{\xi_2^l, \widetilde{\xi_1^{k+1}}} \frac{\mu(v_1^{\xi^{k+1}})}{\mu(v_2^{\xi^l})} \prod_{p=1}^{l-k-1} \left(\delta_{\xi_2^{p+k}, \widetilde{\xi_1^{p+k+1}}} \frac{\mu(v_2^{\xi^{p+k}})}{\mu(v_1^{\xi^{p+k+1}})} \right) \right\} e_{v_0^{\xi^{k+1}}}.$$

Since $f(\xi^k) = s(\xi^{k+1}) = v_0^{\xi^{k+1}}$, we conclude that

$$
\begin{aligned}
&\widetilde{\kappa}_\pi([\xi^1], \cdots, [\xi^n]) \\
&= \left\{ \delta_{\xi_2^l, \widetilde{\xi_1^{k+1}}} \frac{\mu(v_1^{\xi^{k+1}})}{\mu(v_2^{\xi^l})} \prod_{p=1}^{l-k-1} \left(\delta_{\xi_2^{p+k}, \widetilde{\xi_1^{p+k+1}}} \frac{\mu(v_2^{\xi^{p+k}})}{\mu(v_1^{\xi^{p+k+1}})} \right) \right\} \\
&\quad \times \widetilde{\kappa}_\rho([\xi^1], \cdots, [\xi^{k-1}], [\xi^k], [\xi^{l+1}], \cdots, [\xi^n]) \\
&= \left\{ \delta_{\xi_2^l, \widetilde{\xi_1^{k+1}}} \frac{\mu(v_1^{\xi^{k+1}})}{\mu(v_2^{\xi^l})} \prod_{p=1}^{l-k-1} \left(\delta_{\xi_2^{p+k}, \widetilde{\xi_1^{p+k+1}}} \frac{\mu(v_2^{\xi^{p+k}})}{\mu(v_1^{\xi^{p+k+1}})} \right) \right\} \\
&\quad \times \left(\prod_{C=\{c_1, \cdots, c_t\} \in \rho} \left\{ \delta_{\xi_2^{c_t}, \widetilde{\xi_1^{c_1}}} \frac{\mu(v_1^{\xi^{c_1}})}{\mu(v_2^{\xi^{c_t}})} \prod_{p=1}^{t-1} \left(\delta_{\xi_2^{c_p}, \widetilde{\xi_1^{c_{p+1}}}} \frac{\mu(v_2^{\xi^{c_p}})}{\mu(v_1^{\xi^{c_{p+1}}})} \right) \right\} \right) e_{v_0^{\xi^1}} \\
&= \left(\prod_{C=\{c_1, \cdots, c_t\} \in \pi} \left\{ \delta_{\xi_2^{c_t}, \widetilde{\xi_1^{c_1}}} \frac{\mu(v_1^{\xi^{c_1}})}{\mu(v_2^{\xi^{c_t}})} \prod_{p=1}^{t-1} \left(\delta_{\xi_2^{c_p}, \widetilde{\xi_1^{c_{p+1}}}} \frac{\mu(v_2^{\xi^{c_p}})}{\mu(v_1^{\xi^{c_{p+1}}})} \right) \right\} \right) e_{v_0^{\xi^1}} \\
&= S(\pi)([\xi^1 \circ \xi^2 \circ \cdots \circ \xi^n])
\end{aligned}
$$

where the second equality follows from the inductive assumption and equation (22) applied to ρ, and the last equality follows from equation (22) applied to π. □

The proof of Proposition 26 is now immediate.

Proof of Proposition 26. Since $Gr(\Gamma, 0)$ is clearly generated by all the $Gr(\Gamma_w, 0), w \in V_1$, it needs only to be seen that the family $\{Gr(\Gamma_w, 0) : w \in V_1\}$ is free with amalgamation over $P_0(\Gamma, 0)$. This follows from Theorem 25 and Proposition 27. □

Let $\lambda : Gr(\Gamma, 0) \to \mathcal{L}(H(\Gamma, 0))$ also denote the composite of the isomorphism of $Gr(\Gamma)$ with $F(\Gamma)$ and $\lambda : F(\Gamma, 0) \to \mathcal{L}(H(\Gamma, 0))$. Thus, $M(\Gamma, 0) = \lambda(Gr(\Gamma, 0))''$. It now follows fairly easily - see Proposition 4.6 of [KdySnd2008], for instance - that $M(\Gamma_w, 0) \cong \lambda(Gr(\Gamma_w, 0))''$, the content in this statement being that $Gr(\Gamma_w, 0)$ is interpreted as a subalgebra of $Gr(\Gamma, 0)$. We will thus identify $M(\Gamma_w, 0)$ with $\lambda(Gr(\Gamma_w, 0))'' \subseteq M(\Gamma, 0)$.

Now, by general principles, Proposition 26 extends to its von Neumann completions - meaning that

$$M(\Gamma, 0) = *_{P_0(\Gamma,0)}\{M(\Gamma_w, 0) : w \in V_1\}, \tag{23}$$

and similarly, by interchanging the roles of 0 and 1, we have

$$M(\Gamma, 1) = *_{P_0(\Gamma,1)}\{M(\Gamma_v, 1) : v \in V_0\}. \tag{24}$$

4. Graphs with a single odd vertex

In this section we fix the following notation. Let Λ be a graph with at least one edge and a single odd vertex w and even vertices $v_1, \cdots, v_l$. We assume that for $i = 1, \cdots, k$, the vertex v_i is joined to w by $q_i > 0$ edges, while the vertices v_i for $i = k+1, \cdots, l$ are isolated. Thus $k \geq 1$. We also set $\mu^2(v_i) = \alpha_i$ and $\mu^2(w) = \beta$, so that $\beta + \sum_{i=1}^{l} \alpha_i = 1$. Our goal in this section is the explicit determination of the finite von Neumann algebra $M(\Lambda, 0)$.

We begin with a simple observation. The assignments of $Gr(\Gamma)$ or $M(\Gamma)$ to a graph Γ clearly take disjoint unions to (appropriately weighted) direct sums. Thus, if $\tilde{\Lambda}$ denotes the connected component of w in the graph Λ (with normalised restricted μ), then

$$M(\Lambda) \cong \underset{\gamma}{M(\tilde{\Lambda})} \oplus \underset{\alpha_{k+1}}{\mathbb{C}} \oplus \underset{\alpha_{k+2}}{\mathbb{C}} \oplus \cdots \oplus \underset{\alpha_l}{\mathbb{C}} \tag{25}$$

where $\gamma = 1 - \sum_{i=k+1}^{l} \alpha_i$. Note that $M(\Lambda, 1) = M(\tilde{\Lambda}, 1)$. We begin by analysing $M(\Lambda, 1)$ when $k = 1$.

Remark 28. We shall adopt the convention that $LF(1) = L\mathbb{Z}$ whereas $LF(r)$, for $1 < r < \infty$, will be referred to as an interpolated free group factor with finite parameter.

Proposition 29. *Let Ω be a graph with a single even vertex v and single odd vertex w joined by $q > 0$ edges. Let $\mu^2(v) = \alpha$ and $\mu^2(w) = \beta = 1 - \alpha$. Then,*

$$M(\Omega, 1) \cong \begin{cases} LF(q^2) & \text{if } \frac{\alpha}{\beta} > q \\ LF(\frac{2q\alpha}{\beta} - \frac{\alpha^2}{\beta^2}) & \text{if } \frac{1}{q} \leq \frac{\alpha}{\beta} \leq q \\ \underset{1-\frac{\alpha q}{\beta}}{\mathbb{C}} \oplus \underset{\frac{\alpha q}{\beta}}{LF(2 - \frac{1}{q^2})} & \text{if } \frac{\alpha}{\beta} < \frac{1}{q} \end{cases}$$

Proof. Consider $Gr(\Omega, 1)$ generated by q^2 paths of length 2 based at w. Denoting the path $(w \xrightarrow{i} v \xrightarrow{j} w)$ by e_{ij}, the operator-valued (in this case, scalar valued) free cumulant calculation of Proposition 27 implies that the $q \times q$-matrix $X = (([e_{ij}]))$ is a uniformly R-cyclic matrix - in the sense of Definition 10 of [KdySnd2009] - with determining sequence $\alpha_t = \left(\frac{\mu(w)}{\mu(v)}\right)^{t-2}$. Theorem 11 of [KdySnd2009] now implies that X is free Poisson with rate $\frac{\alpha}{\beta q}$. Now, the proof of Proposition 24 of [KdySnd2009] may be imitated to yield the desired result. $\square$

As a consequence, we single out a crisp determination of precisely when $M(\Omega)$ is a factor.

Corollary 30. *Let Ω be as in Proposition 29. Then, $M(\Omega)$ is a factor if and only if $q > 1$ and $\frac{1}{q} \leq \frac{\alpha}{\beta} \leq q$, in which case $M(\Omega) \cong LF(1 + 2q\alpha\beta - \alpha^2 - \beta^2)$.*

Proof. If $M(\Omega)$ is a factor, then, so is $M(\Omega, 1)$ and it follows from Proposition 29 that $\frac{1}{q} \leq \frac{\alpha}{\beta}$. Similarly the factoriality of $M(\Omega, 0)$ and Proposition 29 will imply that $\frac{1}{q} \leq \frac{\beta}{\alpha}$. Thus, $\frac{1}{q} \leq \frac{\alpha}{\beta} \leq q$. To see that $q > 1$, it suffices to observe that if $q = 1$, the already proved inequality shows that $\alpha = \beta$ and then again by Proposition 29, both $M(\Omega, 0)$ and $M(\Omega, 1)$ are $LF(1) \cong L\mathbb{Z}$ so $M(\Omega)$ cannot be a factor.

For the converse, if $q > 1$ and $\frac{1}{q} \leq \frac{\alpha}{\beta} \leq q$, then $q \neq \frac{1}{q}$ and so at least one of the inequalities among $\frac{\alpha}{\beta} \geq \frac{1}{q}$ and $q \geq \frac{\alpha}{\beta}$ must be strict. Hence

$$\begin{aligned} \frac{2q\alpha}{\beta} - \frac{\alpha^2}{\beta^2} &= \frac{\alpha}{\beta}(2q - \frac{\alpha}{\beta}) \\ &> \frac{1}{q}.q \\ &= 1, \end{aligned}$$

and so $M(\Gamma, 1)$ is an interpolated free group factor with finite parameter. Similarly, so is $M(\Gamma, 0)$. By Lemma 19, $M(\Gamma)$ is a factor. Now the corner formula for interpolated free group factors - see [Dyk1994] or [Rdl1994] - implies that $M(\Omega) \cong LF(1 + 2q\alpha\beta - \alpha^2 - \beta^2)$. □

We now wish to analyse $M(\Lambda, 1) = M(\tilde{\Lambda}, 1)$. For this, recall that the weighting, say $\tilde{\mu}$, on $\tilde{\Lambda}$ is given by the normalised restriction of μ. Thus $\tilde{\mu}^2(v_i) = a_i$ for $1 \leq i \leq k$ and $\tilde{\mu}^2(w) = b$, where $\sum_{i=1}^k a_i + b = 1$ and $(\alpha_1 : \cdots : \alpha_k : \beta) = (a_1 : \cdots : a_k : b)$.

Proposition 31. *With the foregoing notation,*

$$M(\tilde{\Lambda}, 1) \cong \begin{cases} LF\left(\sum_{\{i: q_i b < a_i\}} q_i^2 + \sum_{\{i: q_i b \geq a_i\}} \frac{2q_i a_i}{b} - \left(\frac{a_i}{b}\right)^2\right) & \text{if } b \leq \sum_{i=1}^k q_i a_i \\ \underset{1-\sum_i \frac{q_i a_i}{b}}{\mathbb{C}} \oplus \underset{\sum_i \frac{q_i a_i}{b}}{LF(2 - \frac{\sum_i a_i^2}{(\sum_i q_i a_i)^2})} & \text{if } b > \sum_{i=1}^k q_i a_i \end{cases}$$

In particular, $M(\tilde{\Lambda}, 1)$ is an $LF(r)$ for some $r \geq 1$ iff $b \leq \sum_{i=1}^k q_i a_i$.

Proof. By equation (24), we have

$$M(\tilde{\Lambda}, 1) \cong *_{P_0(\tilde{\Lambda}, 1)}\{M(\tilde{\Lambda}_{v_i}, 1) : i = 1, 2, \cdots, k\} \cong * \{M(\tilde{\Lambda}_{v_i}, 1) : i = 1, 2, \cdots, k\},$$

where the second isomorphism holds since $P_0(\tilde{\Lambda}, 1) \cong \mathbb{C}$. Each $M(\tilde{\Lambda}_{v_i}, 1)$ is determined using Proposition 29. Now computations from [Dyk1993] - see Proposition 1.7 - and a little calculation finish the proof. □

Proposition 32. *If Λ has a single odd vertex and at least two edges, then $M(\Lambda) \cong LF(s) \oplus A$, for some finite $s > 1$ and a finite-dimensional abelian A.*

Proof. Notice that $\tilde{\Lambda}$ satisfies the hypotheses of this proposition and in addition, is connected. In view of equation (25), it suffices to prove the proposition for $\tilde{\Lambda}$; in other words, we may assume without loss of generality that Λ is connected.

Hence Theorem 21 is applicable and $M(\Lambda)$ has the form $M \oplus A$ for some II_1 factor M and a finite-dimensional abelian A.

Now Proposition 31 tells us that some corner of $M(\Lambda, 1)$ and hence of $M(\Lambda)$ is an $LF(r)$ for some finite r. On the other hand, the hypothesis that Λ has at least two edges ensures that $M(\Lambda, 1)$ is not commutative and hence $r > 1$. This corner is necessarily a corner of $M \cong LF(s)$ for some finite $s > 1$. $\square$

Corollary 33. *Let Λ be any graph with a single odd vertex and non-empty edge set E. Then,*

$$M(\Lambda, 0) \cong \begin{cases} LF(s) \oplus A & \text{if } |E| > 1 \\ L\mathbb{Z} \oplus A & \text{if } |E| = 1. \end{cases}$$

for some $s > 1$ and finite-dimensional abelian A.

Proof. In case $|E| > 1$, $M(\Lambda, 0)$ is necessarily non-abelian and the desired assertion is a direct consequence of Proposition 32. When $|E| = 1$, observe, as in equation (25), that $M(\Lambda, 0) = M(\tilde{\Lambda}, 0) \oplus A$ for some finite-dimensional abelian A. Now the desired result follows from Proposition 29 applied with Ω being $\tilde{\Lambda}$ with vertex parity reversed. (This is because the q of Proposition 29 is 1 and the parameter occurring in the $LF(\cdot)$ factor is 1 in all the three cases considered there.) $\square$

5. The structure of $M(\Gamma)$

In this section, we determine the structure of $M(\Gamma)$ for any finite, connected, bipartite graph Γ with Perron-Frobenius weighting. The main technical result used in the proof is Theorem 34 which is a consequence of the results in [Dyk2009].

Theorem 34. *Let $M(w), w \in V_1$ be a finite family of tracial von Neumann algebraic probability spaces over a finite-dimensional abelian probability space D. Suppose that each $M(w) \cong LF(r_w) \oplus A(w)$ with $1 \leq r_w < \infty$ and finite-dimensional abelian $A(w)$ and that $M = *_D\{M(w) : w \in V_1\}$ is a factor. Then, M is either an interpolated free group factor with finite parameter or the hyperfinite (II_1) factor.*

The following theorem is one of the main results of this paper.

Theorem 35. *Let a connected graph Γ with at least two edges be equipped with the Perron-Frobenius weighting. Then $M(\Gamma) \cong LF(s)$ for some $1 < s < \infty$.*

Proof. By Corollary 22, $M(\Gamma)$ is a II_1-factor and so, to see that it an interpolated free group factor with finite parameter, it suffices to see that the corner $M(\Gamma, 0)$ is also one.

The hypotheses on Γ ensure that Corollary 33 is applicable to Γ_w for each odd vertex w. Then it follows from equation (23) that the hypotheses of Theorem 34 are satisfied with $D = P_0(\Gamma, 0)$, $M(w) = M(\Gamma_w, 0)$ for $w \in V_1$ and $M = M(\Gamma, 0)$ and so $M(\Gamma, 0)$ is either an interpolated free group factor with finite parameter or the hyperfinite factor.

To conclude the proof, we only need to ensure that $M(\Gamma, 0)$ is not hyperfinite. For this we consider two cases.

Case 1. Suppose some odd vertex w of Γ has degree greater than 1. In this case Corollary 33 shows that $LF(r_w)$ for some $r_w > 1$ is a corner of $M(\Gamma_w, 0)$. A corner of a subalgebra of the hyperfinite factor cannot be $LF(r_w)$ (which is not injective). Hence $M(\Gamma, 0)$ is not hyperfinite.

Case 2. Every odd vertex of Γ has degree 1. Thus Γ is the complete bipartite graph $K(1, n)$ for $n \geq 2$. The Perron-Frobenius weighting on this graph assigns $\frac{1}{\sqrt{2}}$ to the odd vertex and $\frac{1}{\sqrt{2n}}$ to each even vertex. Now, for any odd vertex w, Proposition 29 applied with Ω being Γ_w with reversed vertex parity implies that $M(\Gamma_w, 0) \cong \underset{1-\delta^{-1}}{\mathbb{C}} \oplus \underset{\delta^{-1}}{L\mathbb{Z}}$, where $\delta = \sqrt{n}$. Clearly, $P_0(\Gamma, 0) \cong \mathbb{C}$. Therefore, from equation (23),

$$M(\Gamma, 0) \cong (\underset{1-\delta^{-1}}{\mathbb{C}} \oplus \underset{\delta^{-1}}{L\mathbb{Z}})^{*n} \cong LF(2\sqrt{n} - 1),$$

where the second isomorphism is proved in Corollary 16 of [KdySnd2009]. Since $n \geq 2$, $M(\Gamma, 0)$ is an interpolated free group factor with finite parameter in this case too.

□

The only connected graph to which Theorem 35 does not apply is the graph A_2 which has two vertices joined by a single edge. For completeness, we determine, in the following proposition, the structure of $M(\Gamma)$ in this case.

Proposition 36. *Let Γ be the A_2 graph with a single even vertex v and a single odd vertex w joined by a single edge. Equip Γ with its Perron-Frobenius weighting given by $\mu^2(v) = \frac{1}{2} = \mu^2(w)$. Then $M(\Gamma) \cong M_2(L\mathbb{Z})$.*

Proof. Recall from §1 that with Γ being the A_2 graph, elements of $Gr(\Gamma)$ may be regarded as matrices with rows and columns indexed by the set $\{v, w\}$ and (p, q) entry (with $p, q \in \{v, w\}$) being a linear combination of paths from p to q. We shall write M_{ij} for $e_i M(\Gamma) e_j$ for $0 \leq i, j \leq 1$, where of course $e_0 = e_v$ (resp. $e_1 = e_w$) denotes the projection onto the subspace $\mathcal{H}_0$ (resp. $\mathcal{H}_1$) of $\mathcal{H}$ generated by the set of all paths starting at v (resp. w). Let ξ_n (resp. η_n) be the unique path of length n which starts at v (resp. w). Then, $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, and also (see equation (7)) $\{\{\xi_n\} : n \geq 0\}$ (resp. $\{\{\eta_n\} : n \geq 0\}$) is an orthonormal basis for $\mathcal{H}_0$ (resp. $\mathcal{H}_1$).

Let $x = \lambda(\xi_1) \in M_{01}$. The definition of multiplication in $F(\Gamma)$ shows that $x\{\xi_n\} = 0$ and $x\{\eta_n\} = \{\eta_{n+1}\} + \{\eta_{n-1}\}$ for all $n \geq 1$ (with $\{\eta_{-1}\} = 0$). So, if $w : \mathcal{H}_0 \to \mathcal{H}_1$ is the (obviously unitary) operator defined by $w(\{\xi_n\}) = \{\eta_n\}$, we see that $x = ws$ where $s \in \mathcal{L}(\mathcal{H}_0)$ is the (standard semi-circular) operator given by $s\{\xi_n\} = \{\xi_{n+1}\} + \{\xi_{n-1}\}$. It follows that x is injective (since the Wigner distribution has no atoms).

It follows that if $x = u|x|$ denotes the polar decomposition of x, then $u^*u = e_1$, and similarly one sees that $uu^* = e_0$.

Now if $y \in M_{01}$ is arbitrary, then $y = e_0ye_1$ and we see that $yu^* = (e_0ye_1)(e_0ue_1)^*$ $= e_0ye_1u^*e_0 \in M_{00}$, and hence $y = ye_1 = yu^*u \in M_{00}u$. Arguing similarly, we see that the maps

$$a \mapsto au, a \mapsto u^*a, \text{ and } a \mapsto u^*au$$

define linear isomorphisms of M_{00} onto M_{01}, M_{10} and M_{11} respectively. Finally, it is easy to see that the assignment

$$\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \mapsto \begin{bmatrix} a_{00} & a_{01}u \\ u^*a_{10} & u^*a_{11}u \end{bmatrix}$$

defines an isomorphism of $M_2(M_{00})$ onto $M(\Gamma)$. Since $M_{00} \cong L\mathbb{Z}$ by Proposition 29, the proof is complete. □

6. Application to the GJS construction

In this section we relate our $Gr(\Gamma)$ to the sequence of algebras $Gr_k(P)$ of [GnnJnsShl2008]. Let P be a subfactor planar algebra with finite principal graph $\Gamma = (V, E)$, distinguished vertex $*$ and modulus $\delta > 1$. Thus δ is the Perron-Frobenius eigenvalue of Γ and we let $\mu^2(\cdot)$ be a Perron-Frobenius eigenvector. Let tr be the normalised picture trace on P_n.

Most of the following facts about the tower of algebras

$$(P_{0_+} =)P_0 \subseteq P_1 \subseteq P_2 \subseteq \cdots$$

can all be found in [JnsSnd1997]. For vertices v, w of Γ, we write $\mathcal{P}_n(v, w)$ for the set of paths of length n in Γ beginning at v and ending at w. Similarly we use notation such as $\mathcal{P}_n(v, \cdot)$ for the set of paths of length n in Γ beginning at v.

P_n has a basis given by pairs of paths $(\xi(+), \xi(-))$ in Γ such that $\xi(\pm) \in \mathcal{P}_n(*, v)$ for some vertex $v \in V$. The set $\mathcal{P}_{min}(Z(P_n))$ of minimal central projections of P_n is in natural bijection with $\{v \in V : v = f(\xi) \text{ for some } \xi \in \mathcal{P}_n(*, \cdot)\}$. For such a v, denote the corresponding minimal central projection in P_n by $e(v, n)$ and any minimal projection under $e(v, n)$ by $p(v, n)$. Then, $\{(\xi(+), \xi(-)) : \xi(\pm) \in \mathcal{P}_n(*, v)\}$ are matrix units (meaning $(\xi(+), \xi(-))(\eta(+), \eta(-)) = \delta^{\xi(-)}_{\eta(+)}(\xi(+), \eta(-))$

and $(\xi(+),\xi(-))^* = (\xi(-),\xi(+)))$ for the matrix algebra $e(v,n)P_n$. Further, with $tr(\cdot)$ denoting the normalised picture trace on the planar algebra P, we have $tr(p(v,n)) = \delta^{-n}\frac{\mu^2(v)}{\mu^2(*)}$.

The inclusion of P_n into P_{n+1} is given by

$$\begin{aligned}(26)\,(\xi(+),\xi(-)) \quad &\mapsto \quad \sum_{v\in V}\sum_{\rho(\pm)\in\mathcal{P}_{n+1}(*,v)} \delta^{\xi(+)}_{\rho(+)_{[0,n]}}\delta^{\xi(-)}_{\rho(-)_{[0,n]}}\delta^{\rho(+)_{n+1}}_{\rho(-)_{n+1}}(\rho(+),\rho(-)) \\ &= \quad \sum_{\lambda\in\mathcal{P}_1(f(\xi(\pm),\cdot)} (\xi(+)\circ\lambda,\xi(-)\circ\lambda)\end{aligned}$$

The τ-preserving conditional expectation $P_{n+1}\to P_n$ is given by

$$(\xi(+),\xi(-)) \mapsto \delta^{\xi(+)_{n+1}}_{\xi(-)_{n+1}}\frac{\mu^2(v^{\xi(\pm)}_{n+1})}{\delta\mu^2(v^{\xi(\pm)}_n)}(\xi(+)_{[0,n]},\xi(-)_{[0,n]}). \tag{27}$$

The Jones projection $e_n \in P_n$ for $n \geq 2$ is given by

$$\sum_{v\in V}\sum_{\xi(\pm)\in\mathcal{P}_n(*,v)} \delta^{\xi(+)_{[0,n-2]}}_{\xi(-)_{[0,n-2]}}\delta^{\xi(+)_{n-1}}_{\widetilde{\xi(+)_n}}\delta^{\xi(-)_{n-1}}_{\widetilde{\xi(-)_n}}\frac{\mu(v^{\xi(+)}_{n-1})\mu(v^{\xi(-)}_{n-1})}{\delta\mu^2(v^{\xi(\pm)}_n)}(\xi(+),\xi(-)). \tag{28}$$

In these formulae we have written δ^i_j for the Kronecker delta, for typographical convenience.

Our main observation is that the construction in [GnnJnsShl2008] of $Gr_k(P)$ (after conjugating by suitable powers of the rotation tangle) depends only on the actions of the inclusion, multiplication and right conditional expectation tangles - and not on the left conditional expectations $P_n \to P_{1,n}$. Hence, in principle, '$Gr_k(P)$ depends only on the graph and not on the connection'.

We first need to note that the action of the category epi-TL or $\mathcal{E}$ of §1 on $Gr(\Gamma,*)$ is 'essentially the same' as that of certain annular tangles on $Gr_0(P)$. Consider the full subcategory of $\mathcal{E}$ consisting only of the objects $[0],[2],[4],\cdots$. We will denote this category by $\mathcal{E}_{ev}$. Any morphism in $\mathcal{E}_{ev}$, say an element of $Hom([2n],[2m])$, naturally yields an annular tangle with an internal n-box and an external m-box as in the example in Figure 5 for $n=4, m=1$. This identification of $Hom([2n],[2m])$, composed with the action of annular tangles on planar algebras is seen to yield an $\mathcal{E}_{ev}$ action on $\{P_n\}_{n\geq 0}$. (The tangles in the image of $\mathcal{E}_{ev}$ are what were called 0-good annular tangles in [KdySnd2008].)

We will find it convenient to identify a basis element $(\xi(+),\xi(-))$ of $Gr_0(P)$ with the loop $\xi = \xi(-)\circ\widetilde{\xi(+)}$ based at the $*$ vertex. Equivalently, the loop ξ based at $*$ is identified with $(\widetilde{\xi_{[n,2n]}},\xi_{[0,n]})$.

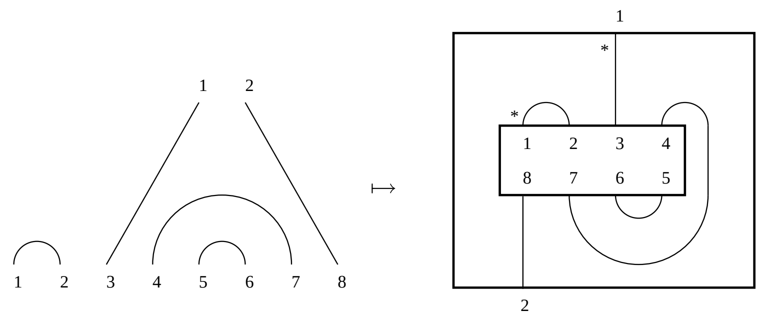

FIGURE 5. Correspondence between $\mathcal{E}_{ev}$ and good Temperley-Lieb tangles

Proposition 37. *The maps* $\{\theta_n : P_{2n}(\Gamma, *) \to P_n\}_{n\geq 0}$ *defined by*

$$\theta_n([\eta]) = \frac{\mu(v_0^\eta)}{\mu(v_n^\eta)}\eta$$

for $[\eta] \in P_{2n}(\Gamma, *)$, *are* $\mathcal{E}_{ev}$*-equivariant.*

Proof. It clearly suffices to verify the intertwining assertion on generators S_i^{2n}. Hence we need to check that for $[\xi] \in P_{2n}(\Gamma, *)$ and $1 \leq i < 2n$ the equality

$$\theta_{n-1}(S_i^{2n}([\eta])) = Z_{S_i^{2n}}(\theta_n([\eta]))$$

holds. There are three cases according as $i < n$, $i = n$ or $i > n$. We will do the first case. The third is similar and the second is easier.

When $i < n$ the annular tangle S_i^{2n} is shown in Figure 6. The dotted lines are meant to indicate a decomposition as the (right) conditional expectation tangle

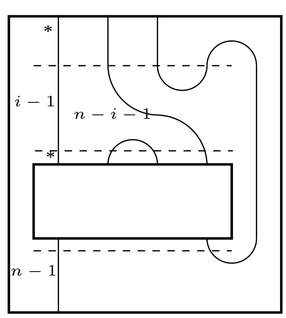

FIGURE 6. The annular tangle S_i^{2n}

applied to the result of post-multiplication with a Temperley-Lieb tangle. The Temperley-Lieb tangle in question here is seen to be the product $E_{i+1}E_{i+1}\cdots E_n$, where $E_t = \delta e_t$. It now follows by induction on $n-i$ using equations (28) and (26) and the multiplication in P_n that $E_{i+1}E_{i+1}\cdots E_n$ is given by

$$\sum_{v\in V}\sum_{\xi(\pm)\in\mathcal{P}_n(*,v)}\delta^{\xi(-)_{[0,i-1]}}_{\xi(+)_{[0,i-1]}}\delta^{\xi(-)_{[i-1,n-2]}}_{\xi(+)_{[i+1,n]}}\delta^{\xi(-)_{n-1}}_{\widetilde{\xi(-)_n}}\delta^{\xi(+)_i}_{\widetilde{\xi(+)_{i+1}}}\frac{\mu(v^{\xi(-)}_{n-1})\mu(v^{\xi(+)}_i)}{\mu(v^{\xi(-)}_n)\mu(v^{\xi(+)}_{i+1})}(\xi(+),\xi(-)).$$

It follows that $Z_{S_i^{2n}}(\eta)$ is given by δ times the conditional expectation onto P_{n-1} of the product

$$\begin{aligned}&(\eta(+),\eta(-))\times\\ &\quad\textstyle\sum_{v\in V}\sum_{\xi(\pm)\in\mathcal{P}_n(*,v)}\delta^{\xi(-)_{[0,i-1]}}_{\xi(+)_{[0,i-1]}}\delta^{\xi(-)_{[i-1,n-2]}}_{\xi(+)_{[i+1,n]}}\delta^{\xi(-)_{n-1}}_{\widetilde{\xi(-)_n}}\delta^{\xi(+)_i}_{\widetilde{\xi(+)_{i+1}}}\frac{\mu(v^{\xi(-)}_{n-1})\mu(v^{\xi(+)}_i)}{\mu(v^{\xi(-)}_n)\mu(v^{\xi(+)}_{i+1})}(\xi(+),\xi(-))\\ &\qquad=\textstyle\sum_{\lambda\in\mathcal{P}_1(v_n^{\eta(-)},\cdot)}\delta^{\eta(-)_i}_{\widetilde{\eta(-)_{i+1}}}\frac{\mu(f(\lambda))\mu(v_i^{\eta(-)})}{\mu(v_n^{\eta(-)})\mu(v_{i+1}^{\eta(-)})}(\eta(+),\eta(-)_{[0,i-1]}\circ\eta(-)_{[i+1,n]}\circ\lambda\circ\widetilde{\lambda}).\end{aligned}$$

Now use equation (27) to conclude that $Z_{S_i^{2n}}(\eta)$ is

$$\begin{aligned}&\sum_{\lambda\in\mathcal{P}_1(v_n^{\eta(-)},\cdot)}\delta^{\eta(-)_i}_{\widetilde{\eta(-)_{i+1}}}\frac{\mu(f(\lambda))\mu(v_i^{\eta(-)})}{\mu(v_n^{\eta(-)})\mu(v_{i+1}^{\eta(-)})}\delta^{\widetilde{\lambda}}_{\eta(+)_n}\frac{\mu^2(v_n^{\eta(-)})}{\mu^2(f(\lambda))}\times\\ &\qquad\qquad(\eta(+)_{[0,n-1]},\eta(-)_{[0,i-1]}\circ\eta(-)_{[i+1,n]}\circ\lambda)\\ &=\delta^{\eta(-)_i}_{\widetilde{\eta(-)_{i+1}}}\frac{\mu(v_n^{\eta(-)})\mu(v_i^{\eta(-)})}{\mu(v_{n-1}^{\eta(+)})\mu(v_{i+1}^{\eta(-)})}(\eta(+)_{[0,n-1]},\eta(-)_{[0,i-1]}\circ\eta(-)_{[i+1,n]}\circ\widetilde{\eta(+)_n})\\ &\qquad=\delta^{\eta_i}_{\widetilde{\eta_{i+1}}}\frac{\mu(v_n^{\eta})\mu(v_i^{\eta})}{\mu(v_{n+1}^{\eta})\mu(v_{i+1}^{\eta})}\ \eta_{[0,i-1]}\circ\eta_{[i+1,2n]}.\end{aligned}$$

Hence,

$$Z_{S_i^{2n}}(\theta_n([\eta]))=\frac{\mu(v_0^\eta)}{\mu(v_n^\eta)}Z_{S_i^{2n}}(\eta)=\delta^{\eta_i}_{\widetilde{\eta_{i+1}}}\frac{\mu(v_0^\eta)\mu(v_i^\eta)}{\mu(v_{n+1}^\eta)\mu(v_{i+1}^\eta)}\ \eta_{[0,i-1]}\circ\eta_{[i+1,2n]}.$$

On the other hand, we have by definition,

$$S_i^{2n}([\eta])=\delta^{\eta_i}_{\widetilde{\eta_{i+1}}}\frac{\mu(v_i^\eta)}{\mu(v_{i+1}^\eta)}[\eta_{[0,i-1]}\circ\eta_{[i+1,2n]}],$$

and thus

$$\theta_{n-1}(S_i^{2n}([\eta]))=\delta^{\eta_i}_{\widetilde{\eta_{i+1}}}\frac{\mu(v_0^\eta)\mu(v_i^\eta)}{\mu(v_{n+1}^\eta)\mu(v_{i+1}^\eta)}\ \eta_{[0,i-1]}\circ\eta_{[i+1,2n]}=Z_{S_i^{2n}}(\theta_n([\eta])),$$

as was to be seen. □

Next, we generalise the path-basis expression for the Jones projections to arbitrary Temperley-Lieb tangles. Let T be a Temperley-Lieb equivalence relation on $\{1,2,\cdots,2n\}$ also identified with a Temperley-Lieb tangle as in the following example. Say $T=\{\{1,2\},\{3,8\},\{4,7\},\{5,6\}\}$. The corresponding tangle is shown in Figure 7. Given such a Temperley-Lieb equivalence relation T we let T_t be the subset of 'through classes', T_u be the subset of 'up classes' and T_d be the set of 'down classes' of T, so that $T=T_t\coprod T_u\coprod T_d$. In this example, $T_t=\{\{3,8\},\{4,7\}\}$, $T_u=\{\{1,2\}\}$ and $T_d=\{\{5,6\}\}$.

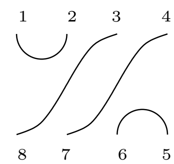

FIGURE 7. The Temperley-Lieb tangle $T = \{\{1,2\},\{3,8\},\{4,7\},\{5,6\}\}$

Proposition 38. *For any Temperley-Lieb equivalence relation T on $\{1,2,\cdots,2n\}$, the element $Z_T(1) \in P_n$ is given by*

$$\sum_{v\in V}\sum_{\xi(\pm)\in\mathcal{P}_n(*,v)}\left(\prod_{\{i,j\}\in T_t:i<j}\delta^{\xi(-)_i}_{\xi(+)_{2n+1-j}}\right)\left(\prod_{\{i,j\}\in T_u:i<j}\delta^{\xi(-)_i}_{\widetilde{\xi(-)_j}}\frac{\mu(v_i^{\xi(-)})}{\mu(v_j^{\xi(-)})}\right)\times$$
$$\left(\prod_{\{i,j\}\in T_d:i>j}\delta^{\xi(+)_{2n+1-i}}_{\widetilde{\xi(+)_{2n+1-j}}}\frac{\mu(v_{2n+1-i}^{\xi(+)})}{\mu(v_{2n+1-j}^{\xi(+)})}\right)(\xi(+),\xi(-)).$$

For instance, for the Temperley-Lieb relation T of Figure 7,

$$Z_T(1) = \sum_{\xi(\pm)\in\mathcal{P}_4(*,\cdot)}\left(\delta^{\xi(-)_3}_{\xi(+)_1}\delta^{\xi(-)_4}_{\xi(+)_2}\right)\left(\delta^{\xi(-)_1}_{\widetilde{\xi(-)_2}}\frac{\mu(v_1^{\xi(-)})}{\mu(v_2^{\xi(-)})}\right)\left(\delta^{\xi(+)_3}_{\widetilde{\xi(+)_4}}\frac{\mu(v_3^{\xi(+)})}{\mu(v_4^{\xi(+)})}\right)(\xi(+),\xi(-)).$$

Proof of Proposition 38. Suppose that $Z_T(1) = \sum_{v\in V}\sum_{\xi(\pm)\in\mathcal{P}_n(*,v)} c_\xi\,(\xi(+),\xi(-))$. Since the $(\xi(+),\xi(-))$ are orthogonal (for the inner product on P_n given by $\langle x,y\rangle = tr(y^*x)$) with $||(\xi(+),\xi(-))||^2 = \delta^{-n}\frac{\mu^2(v_n^{\xi(\pm)})}{\mu^2(*)}$,

$$\begin{aligned}\overline{c_\eta}\delta^{-n}\frac{\mu^2(v_n^{\eta(\pm)})}{\mu^2(*)} &= \langle(\eta(+),\eta(-)),Z_T(1)\rangle\\ &= tr(Z_T(1)^*(\eta(+),\eta(-)))\\ &= \delta^{-n}\times\text{picture trace of } Z_T(1)^*(\eta(+),\eta(-))\\ &= \delta^{-n}\times\text{picture trace of } Z_{T^*}(1)(\eta(+),\eta(-)).\end{aligned}$$

Hence,

$$\overline{c_\eta} = \frac{\mu^2(*)}{\mu^2(v_n^{\xi(\pm)})}\times\text{picture trace of } Z_{T^*}(1)(\eta(+),\eta(-)). \tag{29}$$

We next compute the picture trace of $Z_{T^*}(1)(\eta(+),\eta(-))$. The equivalence relation T^* is the one obtained from T be replacing each i by $2n+1-i$. Regarding T as an element of $Hom([2n],[0])$, there is an associated 0_+-annular tangle, which also we will denote by T. The context and the nature of its arguments should make it clear whether we are referring to the morphism T or the associated Temperley-Lieb tangle T or the associated annular tangle T. Some doodling now shows that

$$\text{picture trace of } Z_{T^*}(1)(\eta(+),\eta(-)) = Z_T((\eta(+),\eta(-))) = Z_T(\eta), \tag{30}$$

where, clearly, T^* is regarded as a Temperley-Lieb tangle and T as an annular tangle.

Finally, notice that Proposition 37 says that

$$T \in Hom([2n],[2m]) \Rightarrow \theta_m \circ T = T \circ \theta_n.$$

When $m = 0$, since $\theta_0 = id_{\mathbb{C}}$, we see that $T = T \circ \theta_n$. i.e.,

$$\begin{aligned} T([\eta]) &= T(\frac{\mu(v_0^\eta)}{\mu(v_n^\eta)}\eta) \\ &= \frac{\mu(v_0^\eta)}{\mu(v_n^\eta)} Z_T(\eta). \end{aligned} \tag{31}$$

However, by Proposition 3, we have

$$T([\eta]) = \frac{\mu(v_n^\eta)}{\mu(v_0^\eta)} \left(\prod_{\{i,j\} \in T: i<j\leq n} \delta^{\eta_i}_{\widetilde{\eta_j}} \frac{\mu(v_i^\eta)}{\mu(v_j^\eta)} \right) \left(\prod_{\{i,j\} \in T: i\leq n<j} \delta^{\eta_i}_{\widetilde{\eta_j}} \right) \times \left(\prod_{\{i,j\} \in T: n<i<j} \delta^{\eta_i}_{\widetilde{\eta_j}} \frac{\mu(v_i^\eta)}{\mu(v_j^\eta)} \right).$$

Putting this together with equations (29),(30) and (31) yields

$$\overline{c_\eta} = \left(\prod_{\{i,j\} \in T: i<j\leq n} \delta^{\eta_i}_{\widetilde{\eta_j}} \frac{\mu(v_i^\eta)}{\mu(v_j^\eta)} \right) \left(\prod_{\{i,j\} \in T: i\leq n<j} \delta^{\eta_i}_{\widetilde{\eta_j}} \right) \left(\prod_{\{i,j\} \in T: n<i<j} \delta^{\eta_i}_{\widetilde{\eta_j}} \frac{\mu(v_i^\eta)}{\mu(v_j^\eta)} \right).$$

Observing that c_η is real and comparing with the statement of the proposition finishes the proof. (Note that when $\{i,j\} \in T$ with $n < i < j$,

$$\delta^{\eta_i}_{\widetilde{\eta_j}} \frac{\mu(v_i^\eta)}{\mu(v_j^\eta)} = \delta^{\eta_i}_{\widetilde{\eta_j}} \frac{\mu(v_{j-1}^\eta)}{\mu(v_{i-1}^\eta)} = \delta^{\widetilde{\xi(+)_{2n+1-i}}}_{\xi(+)_{2n+1-j}} \frac{\mu(v_{2n+1-j}^{\xi(+)})}{\mu(v_{2n+1-i}^{\xi(+)})},$$

which is to be compared with the third product term in the statement.) □

We will now write the structure maps of the algebra $Gr_0(P)$ of [GnnJnsShl2008] in terms of the path bases for the P_n. Recall that the algebra $Gr_0(P) = \oplus_{n=0}^\infty P_n$ is a graded algebra with the multiplication map $\bullet : P_m \otimes P_n \to P_{m+n}$ given by the tangle in Figure 8 below.

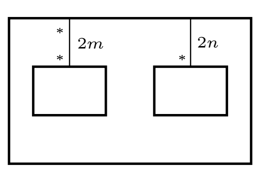

FIGURE 8. Multiplication in $Gr_0(P)$

Proposition 39. *For paths $\xi \in P_m \subseteq Gr_0(P)$ and $\eta \in P_n \subseteq Gr_0(P)$,*

$$\xi \bullet \eta = \frac{\mu(v^{\xi}_{m})\mu(v^{\eta}_{n})}{\mu(v^{\xi\circ\eta}_{m+n})\mu(v^{\eta}_{0})}\ \xi \circ \eta$$

Proof. We will deal with the case $m \geq n$. The other case is similar. The tangle of Figure 8 can be expressed in terms of the inclusion, Temperley-Lieb and multiplication tangles as in Figure 9. Recall that in a tangle picture, a non-negative

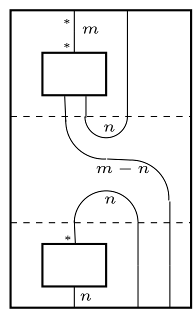

FIGURE 9. Standard tangle expression of tangle in Figure 8

integer t written beside a string indicates a t-cable of the string. We see from this figure that the product of ξ and η in $Gr_0(P)$ is a product of three terms in P_{m+n}, namely, ξ included into P_{m+n}, a Temperley-Lieb tangle and η included into P_{m+n}.

It now follows from Proposition 38 and equation (26) that

$$\begin{aligned}
\xi \bullet \eta \quad = \quad & \left(\sum_{\rho \in \mathcal{P}_m(f(\eta(\pm)),\cdot)} (\eta(+)\circ\rho, \eta(-)\circ\rho) \right) \times \\
& \left\{ \sum_{v \in V} \sum_{\zeta(\pm) \in \mathcal{P}_{m+n}(*,v)} \left(\delta^{\zeta(-)_{[0,m-n]}}_{\zeta(+)_{[2n,m+n]}} \right) \times \right. \\
& \left. \left(\delta^{\zeta(-)^{[m-n,m]}}_{\widetilde{\zeta(-)}_{[m,m+n]}} \frac{\mu(v^{\zeta(-)}_{m})}{\mu(v^{\zeta(-)}_{m+n})} \right) \left(\delta^{\zeta(+)_{[0,n]}}_{\widetilde{\zeta(+)}_{[n,2n]}} \frac{\mu(v^{\zeta(+)}_{n})}{\mu(v^{\zeta(+)}_{2n})} \right) (\zeta(+), \zeta(-)) \right\} \times \\
& \left(\sum_{\lambda \in \mathcal{P}_n(f(\xi(\pm)),\cdot)} (\xi(+)\circ\lambda, \xi(-)\circ\lambda) \right).
\end{aligned}$$

Since the path basis elements multiply as matrix units, given $\lambda, \zeta(\pm)$ and ρ, the product of the terms corresponding to these in the above expression is non-zero only if the following equations hold.

$$\begin{aligned}
\zeta(-) &= \xi(+) \circ \lambda \\
\zeta(+) &= \eta(-) \circ \rho \\
\zeta(-)_{[0,m-n]} &= \zeta(+)_{[2n,m+n]} \\
\zeta(-)_{[m-n,m]} &= \widetilde{\zeta(-)_{[m,m+n]}} \\
\zeta(+)_{[0,n]} &= \widetilde{\zeta(+)_{[n,2n]}}
\end{aligned}$$

A little thought now shows that the following equations are consequences.

$$\begin{aligned}
\zeta(-)_{[0,m]} &= \xi(+) \\
\zeta(+)_{[0,n]} &= \eta(-) \\
\zeta(-)_{[m,m+n]} &= \widetilde{\zeta(-)_{[m-n,m]}} = \widetilde{\xi(+)_{[m-n,m]}} \\
\zeta(+)_{[n,2n]} &= \widetilde{\zeta(+)_{[0,n]}} = \widetilde{\eta(-)} \\
\zeta(+)_{[2n,m+n]} &= \zeta(-)_{[0,m-n]} = \xi(+)_{[0,m-n]} \\
\lambda &= \zeta(-)_{[m,m+n]} = \widetilde{\xi(+)_{[m-n,m]}} \\
\rho &= \zeta(+)_{[n,m+n]} = \widetilde{\eta(-)} \circ \xi(+)_{[0,m-n]}
\end{aligned}$$

Thus, exactly one term is non-zero, which corresponds to $\lambda = \widetilde{\xi(+)_{[m-n,m]}}$, $\rho = \widetilde{\eta(-)} \circ \xi(+)_{[0,m-n]}$, $\zeta(-) = \xi(+) \circ \widetilde{\xi(+)_{[m-n,m]}}$ and $\zeta(+) = \eta(-) \circ \widetilde{\eta(-)} \circ \xi(+)_{[0,m-n]}$. Hence

$$\xi \bullet \eta = \frac{\mu(v_m^{\xi(+)})}{\mu(v_{m+n}^{\xi(+)})} \frac{\mu(v_n^{\eta(-)})}{\mu(v_0^{\eta(-)})} (\eta(+) \circ \widetilde{\eta(-)} \circ \xi(+)_{[0,m-n]}, \xi(-) \circ \widetilde{\xi(+)_{[m-n,m]}})$$

Noting now that $\xi = \xi(-) \circ \widetilde{\xi(+)}$, $\eta = \eta(-) \circ \widetilde{\eta(+)}$ and $\xi \circ \eta = \xi(-) \circ \widetilde{\xi(+)} \circ \eta(-) \circ \widetilde{\eta(+)}$, the proof is finished. $\square$

Proposition 40. *The map $\theta : Gr(\Gamma, *) \to Gr_0(P)$ defined for $[\xi] \in P_{2n}(\Gamma, *)$ by*

$$\theta([\xi]) = \frac{\mu(v_0^\xi)}{\mu(v_n^\xi)} \xi \in P_n$$

and extended linearly is an isomorphism of graded, $*$-probability spaces.

Proof. That θ is a graded, linear isomorphism is clear. Multiplicativity of θ follows from Proposition 39 while $*$-preservation is straightforward. To verify that θ preserves trace, note that by definition of the trace τ in $Gr(\Gamma, *)$, for $[\xi] \in P_{2n}(\Gamma, *)$,

$$\tau([\xi]) = \sum_T \tau_T([\xi]),$$

where the sum is over all Temperley-Lieb equivalence relations T on $\{1, 2, \cdots, 2n\}$ and $\tau_T([\xi])$ is (from the proof of Proposition 5) $\frac{1}{\mu^2(*)} t \circ T([\xi])$ where $t : P_0(\Gamma) \to \mathbb{C}$

is the linear extension of the map taking $[(v)]$ to $\mu^2(v)$. Identifying $P_0(\Gamma,*)$ with $\mathbb{C}$, $\tau_T([\xi]) = T([\xi])$. Equations (30) and (31) now imply that $\tau_T([\xi])$ is $\frac{\mu(v_0^\xi)}{\mu(v_n^\xi)}$ times the picture trace of $Z_{T^*}(1)\xi$. Summing over all Temperley-Lieb equivalence relations gives by definition the trace of $\frac{\mu(v_0^\xi)}{\mu(v_n^\xi)}\xi$ in $Gr_0(P)$, as desired. □

We apply this proposition and Theorem 35 to the GJS construction.

Theorem 41. *Let P be a subfactor planar algebra of finite depth and modulus $\delta > 1$, and M_0 be the factor constructed from P by the construction in* [GnnJnsShl2008]. *Then, $M_0 \cong LF(r)$ for some $1 < r < \infty$.*

Proof. Let Γ be the (finite) principal graph of P equipped with the Perron-Frobenius weighting, so that by Theorem 35, $M(\Gamma)$ is $LF(t)$ for some $1 < t < \infty$. Now Proposition 40 implies that $M(\Gamma,*)$ is isomorphic to M_0 and so $M_0 \cong LF(r)$ for some $1 < r < \infty$. □

Our final result is an analogue of Theorem 41 for the factor M_1 constructed from P.

Theorem 42. *Let P be a subfactor planar algebra of finite depth and modulus $\delta > 1$, and $M_0 \subseteq M_1$ be the subfactor constructed from P by the construction in* [GnnJnsShl2008]. *Then $M_1 \cong LF(s)$ for some $1 < s < \infty$.*

Since the proof is very similar to that of Theorem 41, we will only sketch the proof giving details where it differs from the previous proof. We first recall some preliminaries from [KdySnd2004].

There is an 'operation on tangles' denoted $T \mapsto T^-$ which moves the $*$-region of each of its boxes anti-clockwise by 1 and reverses the shading. There is an associated 'operation on planar algebras' denoted $P \mapsto {}^-P$ where ${}^-P$ is the planar algebra with spaces

$$\begin{aligned} {}^-P_{0_\pm} &= P_{0_\mp} \\ {}^-P_k &= P_k,\ k > 0, \end{aligned}$$

and tangle action defined by $Z_T^{-P} = Z_{T^-}^P$. If P is a subfactor planar algebra, then so is $Q = {}^-P$ and further ${}^-Q$ is isomorphic to P.

Now, given a subfactor planar algebra P, we define a graded, non-commutative probability space ${}^-Gr_1(P)$ associated to P as follows. As a vector space ${}^-Gr_1(P) = \oplus_{n\geq 1} P_n$. The multiplication map $\bullet : P_m \otimes P_n \to P_{m+n-1}$ is defined by the tangle in Figure 10 below. The adjunction map in ${}^-Gr_1(P)$ restricts to the adjunction maps in P_n for each $n \geq 1$. A trace is defined in ${}^-Gr_1(P)$ by letting $\tau(\xi)$ for

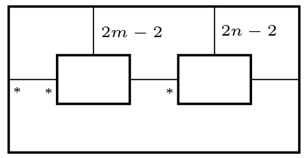

FIGURE 10. Multiplication in ${}^{-}Gr_1(P)$

$\xi \in P_n \subseteq {}^{-}Gr_1(P)$ be the sum over all Temperley-Lieb tangles T of the scalar defined by Figure 11.

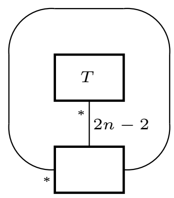

FIGURE 11. T-component of the trace in ${}^{-}Gr_1(P)$

The structure maps of ${}^{-}Gr_1(P)$ are all derived from those of $Gr_1(P)$ (see [GnnJnsShl2008]) using the operation ${}^{-}$. Observe that the vector space underlying both ${}^{-}Gr_1(P)$ and $Gr_1({}^{-}P)$ is the same, namely, $\oplus_{n\geq 1}P_n$. A little thought now yields the following.

Lemma 43. *For any subfactor planar algebra P, the tracial $*$-probability spaces ${}^{-}Gr_1(P)$ and $Gr_1({}^{-}P)$ are isomorphic by the identity map of the underlying vector spaces.* □

Applying Lemma 43 with $Q = {}^{-}P$ in place of P and using that ${}^{-}Q \cong P$ shows that ${}^{-}Gr_1(Q) \cong Gr_1(P)$ as probability spaces. We now proceed towards an analogue of Proposition 40 for ${}^{-}Gr_1(Q)$. Let $\overline{\Gamma}$ denote the principal graph of Q. Since P is of finite depth, so is Q, and thus $\overline{\Gamma}$ is a finite graph. Equip $\overline{\Gamma}$ with its Perron-Frobenius weighting. A basis of Q_n is then given by pairs of paths $(\xi(+), \xi(-))$ in $\overline{\Gamma}$ such that $\xi(\pm)$ are paths of length n in $\overline{\Gamma}$ beginning at its $*$ and having the same end-point. Again, we identify the basis element $(\xi(+), \xi(-))$ with the loop $\xi(-) \circ \widetilde{\xi(+)}$ based at $*$.

Observe that the 0^{th}-graded piece of ${}^{-}Gr_1(Q)$ can be identified as an algebra with Q_1^{op}. In particular, each vertex v in $\overline{\Gamma}$ at distance 1 from its $*$ vertex gives a minimal central projection $f(v, 1)$ in $[{}^{-}Gr_1(Q)]_1 = Q_1^{op}$ and we denote by $q(v, 1)$ any minimal projection of Q_1^{op} lying under $f(v, 1)$. A choice of $q(v, 1)$ is the matrix unit $(\nu, \nu) \in Q_1^{op}$ where ν is any path of length 1 in $\overline{\Gamma}$ from $*$ to v. We fix this choice.

The following proposition expresses the multiplication of ${}^{-}Gr_1(Q)$ in terms of its path basis.

Proposition 44. *For paths $\xi \in Q_m \subseteq {}^{-}Gr_1(Q)$ and $\eta \in P_n \subseteq {}^{-}Gr_1(Q)$,*

$$\xi \bullet \eta = \delta^{\xi_{2m}}_{\widetilde{\eta_1}} \frac{\mu(v^{\xi}_m)\mu(v^{\eta}_n)}{\mu(v^{\xi\circ\eta}_{m+n-1})\mu(v^{\eta}_1)} \; \xi_{[0,2m-1]} \circ \eta_{[1,2n]}$$

Sketch of proof. Suppose that $m \geq n$. The key fact is that the tangle of Figure 10 is expressible in terms of the inclusion, Temperley-Lieb and multiplication tangles as in Figure 12. We omit the rest of the proof which is very similar to that of

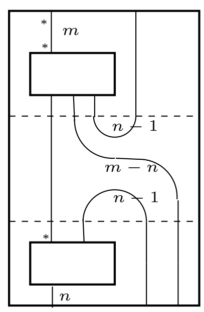

FIGURE 12. Multiplication of ${}^{-}Gr_1(Q)$ in terms of standard tangles

Proposition 39. □

It follows from Proposition 44 that a basis of $q(v,1)({}^{-}Gr_1(Q))q(v,1)$ is given by all paths of the form $\nu \circ \xi \circ \widetilde{\nu}$ where ξ ranges over all paths in $\overline{\Gamma}$ from v to v. This is suggestive of the following key isomorphism which is the analogue of Proposition 40 and whose proof is similar (and omitted).

Proposition 45. *The map $\theta : Gr(\overline{\Gamma}, v) \rightarrow q(v,1)({}^{-}Gr_1(Q))q(v,1)$ defined for $[\xi] \in P_{2n}(\overline{\Gamma}, v)$ by*

$$\theta([\xi]) = \frac{\mu(v^{\xi}_0)}{\mu(v^{\xi}_n)} \nu \circ \xi \circ \widetilde{\nu} \in Q_{n+1}$$

and extended linearly is an isomorphism of graded, $*$-probability spaces. □

We conclude with the proof of Theorem 42.

Proof of Theorem 42. From Proposition 45 and the isomorphism of ${}^{-}Gr_1(Q)$ with $Gr_1(P)$, it follows by completing that some corner of M_1 is isomorphic to $M(\overline{\Gamma}, v)$ - a corner of $M(\overline{\Gamma})$. Since $M(\overline{\Gamma})$ is an interpolated free group factor with finite parameter by Theorem 35, the proof is complete. □

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The Institute of Mathematical Sciences, Chennai, India
E-mail address: vijay@imsc.res.in

The Institute of Mathematical Sciences, Chennai, India
E-mail address: sunder@imsc.res.in