Planar algebras, random matrices and the $O(n)$ model

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Introduction

Moments of large random matrices are related
with the enumeration of planar maps

Random matrices allow to define natural traces on planar algebras.

Random matrices can be used to study such traces
and solve the related combinatorial questions

Outline

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Random matrices and the enumeration of planar maps

Planar algebras

Matrix models generalization

One random matrix

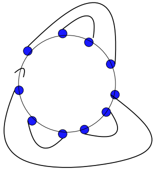
Let X^N be a GUE matrix, that is an $N \times N$Hermitian matrix with i.i.d complex entries with Gaussian law :

$$d\mu^N(X^N) = \frac{1}{Z^N} e^{-\frac{N}{2}\mathrm{tr}(X_N X_N^*)} dX_N$$

with $dX_N = \prod_{i \le j} d\Re X_{ij} \prod_{i<j} d\Im X_{ij}$. Then, Wigner (58) showed that

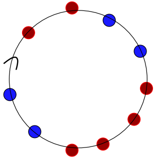
$$\lim_{N\to\infty} \int \frac{1}{N}\mathrm{tr}((X^N)^k) d\mu^N(X^N) = C_{\frac{k}{2}}$$

with $C_{k/2}$ the number of pair non-crossing partitions of k elements.

$C_{\frac{k}{2}} = \sharp\{$  $\}$

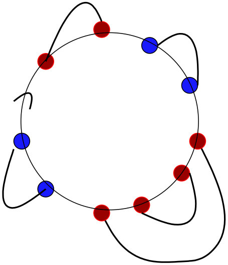
Several independent random matrices

Let $X_i^N, 1 \leq i \leq r$ be independent copies of the GUE. Associate (bijectively) to a word $q(x_1, \ldots, x_r) = x_{i_1} \cdots x_{i_m}, 1 \leq i_k \leq r$ a star of type q=oriented circle with dots of color i_1, i_2, $\ldots i_d$, ordered clockwise, the first dot being marked. Here $q(X) = {X_1}^2 {X_2}^2 {X_1}^4 {X_2}^2$.



Then, Voiculescu (84)

$$\lim_{N\to\infty} E[\frac{1}{N}\mathrm{tr}(q(X))] = \sharp\{$$



$$\}$$

Matrix models

Let $V(X_1,\ldots,X_m)=\sum_{i=1}^r t_i q_i(X_1,\ldots,X_m)$ and consider the m random matrices with law

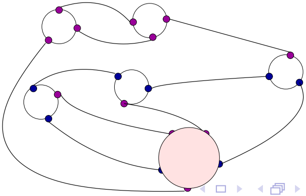
$$d\mu_V^N(X_1,\ldots,X_m)=\frac{1}{Z_V^N}1_{\|X_i\|_\infty\le L}e^{N\mathrm{tr}V(X_1,\ldots,X_m)-\frac{N}{2}\mathrm{tr}(\sum X_i^2)}\prod dX_i.$$

Then,Brézin-Parisi-Itzykson-Zuber (84), Guionnet–Maurel-Segala (06), for L large enough and t_i's small enough

$$\lim_{N\to\infty}\int\frac{1}{N}\mathrm{tr}(q(X))d\mu_V^N(X)=\sum_{k_1,\ldots,k_r\ge 0}\prod_{i=1}^r\frac{t_i^{k_i}}{k_i!}\sharp\{\text{number of planar maps}$$

$$\text{with a star of type q and } k_i \text{ stars of type } q_i\}$$

$$\begin{aligned} q(X) &= X_1^2X_2X_1X_2,\\ q_1(X) &= X_1^3,\ k_1=2,\\ q_2(X) &= X_1X_2^2, k_2=3 \end{aligned}$$



Temporary conclusion

Moments of random matrices are given by
number of planar diagrams connecting stars indexed by monomials.

Considering instead of monomials special polynomials
corresponding to embeddings of planar algebras allows to replace stars
by elements of planar algebras and thus yields traces on the latter.

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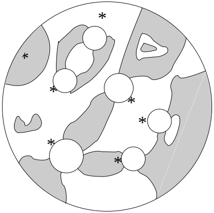
Planar algebras

Matrix models generalization

Planar algebras, Jones (99)

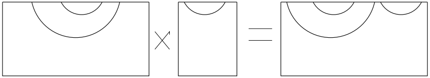
Elements of a planar algebra are discs embedded in a disc connected by non intersecting strings and with boundary points.

They can be endowed with diverse multiplications and tracial states.



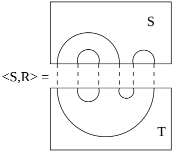
Temperley-Lieb algebra

Temperley-Lieb algebras are exemples of planar algebras, with no internal discs. We can endow this set with the multiplication :



and the trace given by

$$\tau(S) = \sum_{R \in \mathrm{TL}} \delta^{\sharp \text{loops in} <S,R>}$$



Theorem (G-Jones-Shlyakhtenko 07')

Take $\delta \in I := \{2\cos(\frac{\pi}{n})\}_{n \geq 4} \cup]2, \infty[$

-τ is a tracial state, as a limit of matrix (or free var.) models.

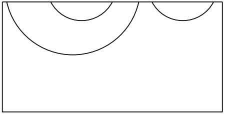
-The corresponding von Neumann algebra is a factor. A tower of factors with index δ^2 can be built.

Random matrix construction

Temperley-Lieb algebra can be embedded in the ring of polynomials in non-commutative indeterminates.

Let $\delta = n \in \mathbb{N}$. For a TL element B, we denote $p \overset{B}{\sim} \ell$ if a string joins the pth dot with the ℓth dot in B, then we associate to B with k strings the polynomial $\sum_{i_j = i_p \text{ if } j \overset{B}{\sim} p} X_{i_1} \cdots X_{i_{2k}}$.

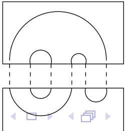
$$q_B(X) = \sum_{i,j,k=1}^{n} X_i X_j X_j X_i X_k X_k \Leftrightarrow$$



With μ^N the law of n independent GUE matrices,

$$\lim_{N\to\infty} \int \frac{1}{N} \mathrm{tr}\left(q_B(X)\right) \mu^N(dX) = \sum_S n^{\sharp \text{loops in } <S,B>} = \tau(B)$$

where we sum over all TL elements with the same number of strings



More δ's : The planar algebra of a bipartite graph

Let $\Gamma = (V, E)$ be a bipartite graph with an eigenvalue δ and a Perron-Frobenius eigenvector μ. $\mu(v) \geq 0, v \in V$, $\delta \in \{2\cos(\frac{\pi}{n})\}_{n\geq 4} \cup]2, \infty[$

A Temperley-Lieb element B is given a set of loops $L_B = \{w = e_1 \cdots e_k, e_i \in E\}$ in Γ so that $e_n = e_\ell^o$ if a string connects $\{n, \ell\}$ in B and $s(e_1) = t(e_k)$. The weight of w is, if $\sigma(e) := \sqrt{\frac{\mu(t(e))}{\mu(s(e))}}$ with $e = (s(e), t(e))$,

$$\sigma_B(w) = \sigma(e_{i_1}) \cdots \sigma(e_{i_n}) \text{ if } e_{i_k} = e_{j_k}^o \text{ whenever } i_k \overset{B}{\sim} j_k \text{ and } \; i_k < j_k,$$

Then we associate to B

$$w_B = \sum_{w \in L_B} \sigma_B(w) w.$$

Exemple : $\delta = n$, $\Gamma = A_n$ the graph with two vertices and n edges, $\mu(v) = 1$, $w_B = \sum_{w \in L_B} w$.

General δ's : Random matrix model

Let $\Gamma = (V, E)$ be a bipartite graph with an eigenvalue δ and a Perron-Frobenius eigenvector μ. Let X_e^N be a $N_{s(e)} \times N_{t(e)}$ matrix with i.i.d centered Gaussian entries with variance $(N_{s(e)} N_{t(e)})^{-\frac{1}{2}}$ so that

$$\lim_{N\to\infty} \frac{N_v}{N} = \mu(v).$$

Then, for any Temperley-Lieb element B,

$$\begin{aligned}\tau(B) &:= \sum_{S \in \mathrm{TL}} \delta^{\sharp \text{loops in} <S,B>} \\ &= \lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N_v} \mathrm{tr}\left(\sum_{\substack{w=e_1\cdots e_k \in L_B \\ s(e_1)=v}} \sigma_B(w) X_{e_1}^N \cdots X_{e_k}^N\right)\right]\end{aligned}$$

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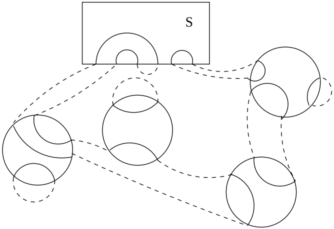
Matrix model generalization

Let $S_1, \ldots, S_n$ be Temperley-Lieb elements. Let $\beta_1, \cdots, \beta_n$ be small real numbers and for any Temperley-Lieb element S

$$\mathrm{Tr}_\beta(S) = \sum_{n_i \geq 0} \sum \prod_{1 \leq i \leq n} \frac{\beta_i^{n_i}}{n_i!} \delta^{\sharp \mathrm{loops}}$$

where we sum over all connected planar diagrams with n_i diagrams S_i and one diagram S.

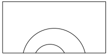


Theorem (G-Jones-Shlyakhtenko-Zinn Justin 10')

Take $\delta \in I := \{2\cos(\frac{\pi}{n})\}_{n \geq 4} \cup]2, \infty[$

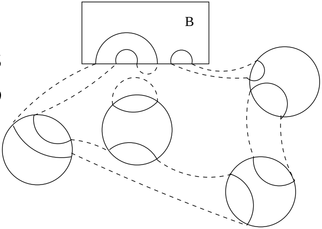
Tr_β is a tracial state, as a limit of matrix (or free probability) models.

A well known example from Physics : the $O(n)$ model

Let $S =$ . Then the $O(n)$ model corresponds to the trace

$$\tau_\beta(B) = \sum n^{\sharp \text{loops}} \frac{\beta^p}{p!}$$

where the sum runs over planar maps with p tangles S and one B

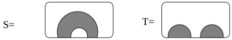


Let $\mu_\beta^N(dX) = \frac{1}{Z_\beta^N} 1_{\|X_i\|_\infty \le L} e^{N \mathrm{tr}(\beta(\sum_{i=1}^n X_i^2)^2 - \frac{1}{2}\sum_{i=1}^n X_i^2)} \prod dX_i$

$$\tau_\beta(B) = \lim_{N\to\infty} \int \frac{1}{N} \mathrm{tr}\Big(\sum_{i_j = i_\ell \text{ if } j \overset{B}{\sim} \ell} X_{i_1} \cdots X_{i_k} \Big) \mu_\beta^N(dX)$$

Planar algebras of bipartite graphs and the shaded $O(\delta)$ model

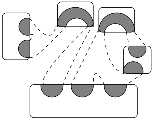
As for planar algebras, it is convenient to shade tangles. The shaded $O(n)$ model corresponds to shaded tangles



and the trace given for a shaded TL element B by

$$\tau_{\beta,\kappa}(B) = \sum \delta^{\sharp \text{loops}} \frac{\beta^k}{k!} \frac{\kappa^p}{p!}$$

with the sum over planar maps built over 1 tangle B, k tangles T and p tangles S so that the shading agrees.



Matrix model for the shaded $O(n)$ model

Let $\delta \in \{2\cos(\frac{\pi}{n})\}_{n\geq 4} \cup]2,\infty[$ and $\Gamma = (V_+ \cup V_-, E)$ be a bipartite graph with eigenvalue δ and Perron-Frobenius eigenvector μ. Denoting E_- (resp. E_+) the edges starting in V_- (resp. V_+) we consider

$$\mu^N_{\beta,\kappa}(dX) = \frac{1_{\|X_e\|_\infty \leq L}}{Z^N_{\beta,\kappa}} e^{N\mathrm{tr}\left(\sum_{v\in V}\mu(v)(\beta 1_{v\in V_+}+\kappa 1_{v\in V_-})\left(\sum_{e:s(e)=v}\sigma(e)X_eX_e^*\right)^2\right)}$$

$$\prod_e e^{-\frac{N}{2}(\mu(s(e))\mu(t(e)))^{\frac{1}{2}}\mathrm{tr}(X_eX_e^*)} dX_e$$

Theorem (G-Jones-Shlyakhtenko-Zinn Justin 10')

Then, for L large enough, β,κ small enough

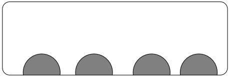
$$\lim_{N\to\infty} \frac{1}{N\mu(v)} \int tr\Big(\sum_{\substack{w=e_1\cdots e_k\in L_B \\ s(e_1)=v}} \sigma(w) X_{e_1}\cdots X_{e_k}\Big)\mu^N_{\beta,\kappa}(dX) = \tau_{\beta,\kappa}(B)$$

Solving the matrix model

For a shaded TL element B, let $\tau_{\beta,\kappa}$ be the trace given by by

$$\tau_{\beta,\kappa}(B) = \sum \delta^{\sharp \text{loops}} \frac{\beta^k}{k!} \frac{\kappa^p}{p!}$$

Theorem (G-Jones-Shlyakhtenko-Zinn Justin 10')

For $\delta \in \{2\cos(\frac{\pi}{n})\}_{n\geq 4} \cup]2,\infty[$ and a Temperley-Lieb element B whih is a product of cups,  *, there exists an explicit formula for $\tau_{\beta,\kappa}(B)$.*

This computation is done thanks to matrix models formulation and allows to retrieve Bernadi–Bousquet-Melou (09) computation for a continuum of δ's and thus for all δ's.

Idea of the proof

With G_v $N_v \times N_v$ independent matrices from the GUE and X_e $N_{s(e)} \times N_{t(e)}$ matrices with covariance $(N\mu(s(e))\mu(t(e)))^{-\frac{1}{2}}$ under μ^N, with $\alpha(v) = 1_{v\in V_+}\sqrt{2\beta} + 1_{v\in V_-}\sqrt{2\kappa}$,

$$
\begin{aligned}
Z^N_{\beta,\kappa} &= \int e^{N\mathrm{tr}\left(\sum_{v\in V}\mu(v)(\beta 1_{v\in V_+}+\kappa 1_{v\in V_-})\left(\sum_{e:s(e)=v}\sigma(e)X_eX_e^*\right)^2\right)}\mu^N(dX)\\
&= \int e^{N\mathrm{tr}\left(\sum_{v\in V}\alpha(v)G_v\left(\sum_{e:s(e)=v}\sigma(e)X_eX_e^*\right)\right)}\mu^N(dX,dG)\\
&= \int \prod_{e\in E_+} e^{-\mathrm{tr}\otimes\mathrm{tr}(\log(I+\alpha(s(e))I\otimes G_{s(e)}+\alpha(t(e))G_{t(e)}\otimes I))}\mu^N(dG)\\
&= \int \prod_{e\in E_+} e^{\sum_{i=1}^{N_{s(e)}}\sum_{j=1}^{N_{t(e)}}\log(1-\sqrt{2\beta}\lambda_i^e-\sqrt{2\kappa}\eta_j^e)}\\
&\qquad \Delta(\eta^e)^2\Delta(\lambda^e)^2 e^{-\frac{N_{s(e)}}{2}\sum_{i=1}^{N_{s(e)}}(\lambda_i^e)^2-\frac{N_{t(e)}}{2}\sum_{j=1}^{N_{t(e)}}(\eta_j^e)^2}d\lambda^e d\eta^e.
\end{aligned}
$$

Conclusion

- Random matrices allow to define natural traces on Temperley-Lieb algebras, and more generally on any planar algebras.
- Planar algebras of a bipartite graph give a natural construction of such laws for any possible $\delta \in \{2\cos(\frac{\pi}{n})\}_{n\geq 4} \cup]2, \infty[$.
- Thus, this point of view entitles to use random matrix models to compute the corresponding generating functions for a continuum of values of δ's and to make rigorous various analytic extension in the physics litterature.