3. Ramanujan's Notebooks

The history of the notebooks, in brief, is the following: Ramanujan had noted down the results of his researches, without proofs, (as in *A Synopsis of Elementary Results, a book on pure Mathematics,* by G.S. Carr), in three notebooks, between the years 1903 - 1914, before he left for England. These were the notebooks which he showed to his benefactors to convince them about his abilities as a mathematician. The results in these notebooks were organized by him. The first notebook has 16 chapters in 134 pages. The second is a revised, enlarged version of the first, containing 21 chapters in 252 pages. The third notebook contains 33 pages of unorganized material (included at the end of volume 2 of the facsimile edition [VII]). Ramanujan took these notebooks with him to Cambridge. But, in one of his letters to a friend, he wrote that he had no time to look into them and most probably he did not put them to use during his five year stay abroad.

The Preface to the *Collected Papers of Srinivasa Ramanujan* ([III], p.ix) contains the following appraisal of the work in the notebooks:

> *There is still a large mass of unpublished material. None of the contents of Ramanujan's notebooks has been printed, unless incorporated in later papers, except that one chapter, on generalized hypergeometric series, was analyzed by Hardy* [37] *in the Proceedings of the Cambridge Philosophical Society. This chapter is sufficient to show that, while the notebooks are naturally unequal in quality, they contain much which should certainly be published. It would be a very formidable task to work through them systematically, select particular passages, and edit these with adequate comment, and it is impossible to print the notebooks as they stand without further monetary assistance. The singular quality of Ramanujan's work and the romance which surrounds his career, encourage us to hope that this volume may enjoy sufficient success to make possible the publication of another.*

Hardy tried to persuade the University of Madras to undertake such a task and in turn, Prof. G.N. Watson was requested by the Madras University, in 1931, to edit the notebooks in a suitable form for publication. This was a formidable task since the notebooks contained about 4000 theorems.

Prof. Watson gave a lecture [39] on Ramanujan's notebooks at the London Mathematical Society in 1931, and published several papers between 1928–36, motivated by Ramanujan's work. Watson's Presidential address to the London Mathematical Society, in 1935, was on *mock-theta functions*, discovered by Ramanujan during the period when he was terminally ill (March 1919 – April 1920) after his return to India. Watson undertook the task of editing the notebooks with Prof. B.M. Wilson and

> *they focussed their attention on the second notebook. Chapters 2–13 were to be edited by Wilson, and Watson was to examine chapters 14–21. Unfortunately, Wilson passed away prematurely in 1935 at the age of 38.* (Berndt [40]).

This possibly put an end to the joint effort of Watson and Wilson to edit the notebooks.

In 1957, with monetary assistance from Sir Dadabai Naoroji Trust, at the instance of Prof. Homi J. Bhabha and Prof. K. Chandrasekaran of the Tata Institute of Fundamental Research, the Tata Institute published a facsimile edition of the notebooks of Ramanujan in two volumes. The reproduction is thoroughly faithful to the original. About a thousand copies were made then and in the centenary year of Ramanujan's birth, 1987, Springer - Narosa reprinted the same.

The formidable task of truly editing the notebooks –*viz.* to either prove each of the results or provide references to the literature where the proofs may be found – which was started by G. N. Watson and B. M. Wilson but never completed, was taken up in right earnest by Prof. Bruce C. Berndt, who has been publishing a series of papers, since 1981, by himself or in collaboration with others, including Indian Mathematicians Padmini T. Joshi, C. Adiga and S. Bhargava. He has also utilized the earlier work of G.N. Watson and B.M. Wilson on the notebooks with the Trinity College. This dedicated work of Berndt, published by Springer-Verlag [VIII], is now available in five parts, the first of which appeared in 1985 and the fifth, in 1997.

At the beginning of his work, in Part I, Berndt [34] points out:

> *The notebooks were originally intended primarily for Ramanujan's own personal use and not for publication ... Some of Ramanujan's incorrect 'theorems' in number theory found in his letters to*

> *Hardy have been well publicized. Thus, perhaps, some think that Ramanujan was prone to making errors. However, such thinking is erroneous. The notebooks contain scattered errors. Especially if one takes into account the roughly hewn nature of the material and his frequently formal arguments, Ramanujan's accuracy is amazing.*

Resurgence of interest

After Ramanujan returned to India, though he was seriously ill and bedridden for most of the time, he was working continuously. His life-long habit had been to work on a slate and transfer the final results to the notebook or sheets of paper. His brother-in-law has stated [59] that *a vast quantity of papers containing his notes was handed over to the University after his death.* Three days after Ramanujan died, his brother S. Lakshminarasimhan, in a letter to Prof. Hardy, dated 29th April 1920, informing him about his demise, wrote :

> *All his Mss that were in his trunk were handed over on the day of his death to Mr. Ramachandra Row's son in law, since the former is at Nilgiris. Not only those but also the journals, magazines - all he possessed except the books which he had, were taken away by them.*

In a letter to Prof. Hardy, dated 3 Dec. 1920, Mr. R. Ramachandra Row, wrote:

> *Ramanujan's M.S.S whatever they are are with me and will be handed over intact to the University of Madras who I understand is already in correspondence with you regarding the methods of publication.*

These papers reached Prof. Hardy, who should have passed them on to Prof. G.N. Watson and hence to their final destination – viz. the estate of Prof. Watson, in the underground cellar of Trinity College – after he wrote several papers inspired Ramanujan's results, until their discovery in 1976 by George E. Andrews.

Prof. Watson died in Feb. 1965. A decade later, in the spring of 1976, when Prof. George Andrews [53] of the Pennsylvania State University was going through the estate of Watson, he discovered a box of papers.

> *The most interesting item in this box was a manuscript of more*

> *than one hundred pages in Ramanujan's distinctive handwriting which contains over six hundred mathematical formulae listed one after the other without proof. It is my contention that this manuscript, or notebook, was written during the last year of Ramanujan's life after his return to India from England. My evidence for this assertion is all indirect; in the words of Stephen Leacock, 'It is what we call circumstantial evidence — the same that people are hanged for' .*

These are now referred to as the contents of the *Lost* notebook of Ramanujan. Andrews assessed ([101], p.xii):

> *While it is impossible to categorize the various formulas completely, a rough approximation of its contents is the following:*
>
> *q-series and related topics including mock θ-functions: 60%*
> *Modular equations and relation, singular moduli: 30%*
> *Integrals, Dirichlet series, congruences, asymptotics, misc.: 10%*
>
> *I give only this rough break down because of the chaotic nature of this manuscript. On many pages there are fragments of numerical computations and infinite series running off in all directions. It may be possible eventually to make sense out of some of these pages; if so, the above percentage breakdown may change. ...*
> *In any event it is clear that q-series investigations make up the bulk of the work in in the Lost Notebook. Indeed I count about 380 formulae that I consider to belong more to q-series than to modular relations or other topics.*

Sixty-seven years after the death of Ramanujan, due to the discovery of the 'lost' notebook by Prof. George Andrews, in 1976, and the editing of the three Notebooks of Ramanujan by Prof. Bruce Berndt, there has been a resurgence of interest in the work of Ramanujan. These notebooks of Ramanujan have formed the basis for numerous papers by many mathematicians, who gave proofs of the theorems and conjectures of Ramanujan obtained by him through his intuition and sheer brilliance. This is a singular and unparalleled phenomenon in the annals of mathematics. In the centenary year of his birth, 1987, several conferences were held in many countries to focus attention on his work and the recent developments. Mystery still surrounds *some of his* *work, mostly in the theory of elliptic functions*, wherein

> *it is not possible, after all the work of Watson and Mordell, to draw the line between what he may have picked up somehow and what he must have found for himself.* ([IV], p.10).

Hardy [60] regretted that he could easily have but did not find out all the details from Ramanujan himself since he

> *was quite able and willing to give a straight answer to a straight question, and not in the least disposed to make a mystery of his achievements.*

However, Hardy [60] argues that Ramanujan

> *was not particularly interested in his own history or psychology; he was a mathematician anxious to get on with the job. And after all I too was a mathematician, and a mathematician meeting Ramanujan had more interesting things to think about than historical research. It seemed ridiculous to worry him about how he had found this or that known theorem, when he was showing me half a dozen new ones almost every day.*

A Page from the second Notebook

To provide a feeling for Ramanujan's notebooks, we reproduce here a page [61] from his second Notebook, Chapter XVIII. There is no particular reason for choosing this page except that it contains an entry which has become a 'folklore' which illustrates the attachment Ramanujan had to numbers prompting Littlewood to state that to Ramanujan every number is a *personal friend.*

There are 2 parts shown in this page. The first part is concerning the geometrical construction of a square whose area is equal to that of a given circle. This is a rare piece of Ramanujan's work in geometry, since he did very little work in geometry and perhaps did not have much interest in it. Several attempts on this problem have been made from early times by (ancient Greek and Hindu) mathematicians and it has been shown that it is impossible

24. To construct a square equal to a given circle.

Let O be the centre and PR any diameter.

Bisect OP at H and trisect OR at T. Draw TQ perp to OP.

Draw RS = TQ. Join PS.

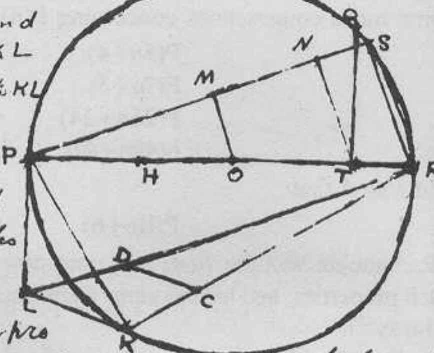
Draw OM & TN ∥ to RS.

Draw PK = PM; & PL = MN and perp to OP. Join RL, RK & KL.

Cut off RC = RH. Draw CD ∥ to KL.

Then $RD^2 = \odot PQR$.

N.B. RD is $\frac{1}{100}$th of an inch greater than the true length if the given ⊙ is 14 Sq. miles in area.



Cor. 1. One of the two mean proportionals between a side of an equilateral triangle inscribed in the ⊙ and the length PS is only less ~~than~~ 30000 th part of it than the true length.

Cor. 2. The app. length got by assuming $\pi = \sqrt[4]{97\frac{1}{2} - \frac{1}{11}}$ is $\frac{1}{100}$th of an inch less than the true length if the ⊙ is a million square miles in area.

ii. $\left\{6n^2 + (3n^3 - n)\right\}^3 + \left\{6n^2 - (3n^3 - n)\right\}^3 = \left\{6n^2(3n^2+1)\right\}^2$

iii. $\left\{m^7 - 3m^4(1+p) + m(3\overline{1+p}^2 - 1)\right\}^3$

$+ \left\{2m^6 - 3m^3(1+2p) + (1+3p+3p^2)\right\}^3$

$+ \left\{m^6 - (1+3p+3p^2)\right\}^3 = \left\{m^7 - 3m^4 p + m(3p^2 - 1)\right\}^3$

ex. $(11\frac{1}{2})^3 + (\frac{1}{2})^3 = 39^2$; $(3 - \frac{1}{105})^3 + (\frac{1}{105})^3 = (5\frac{6}{35})^2$.

$(3\frac{1}{7})^3 - (\frac{1}{7})^3 = (5\frac{4}{7})^2$; $(3\frac{1}{102})^3 - (\frac{1}{102})^3 = (5\frac{23}{102})^2$.

$3^3 + 4^3 + 5^3 = 6^3$; $1^3 + 12^3 = 9^3 + 10^3$; $1^3 + 75^3 = [illegible]$

$3^3 + 509^3 + 34^6 = 1188^3$; $18^3 + 19^3 + 21^3 = 28^3$

$7^3 + 14^3 + 17^3 = 20^3$; $19^3 + 60^3 + 69^3 = 82^3$; $15^3 + 82^3 + 89^3$

$= 108^3$; $3^3 + 36^3 + 37^3 = 46^3$; $1^3 + 135^3 + 138^3$ 172^3.

Fig. caption: The last three lines of p. 224 and the p. 225 from the second notebook of Ramanujan, chapter XVIII. ([61]).

to *square the circle*, in the Euclidian sense[1]. Hardy has commented that he found in Ramanujan's collection of books in Cambridge some books by 'quacks' on this theme.

Ramanujan published a paper [62] entitled, "*Squaring the circle*" in the Jour. of the I.M.S. in 1913, in which he gives a geometrical construction for finding the length of the side of a square whose area equals that of the circle. He also reproduced this and another geometrical construction for π in his later paper [63] on "*Modular equations and approximations to* π". In this paper, he deduced several formulae[2] for π like:

$$\frac{63}{25}\frac{17+15\sqrt{5}}{7+15\sqrt{5}} = 3.14159265380...$$

$$\pi = \frac{24}{\sqrt{142}} log\left\{\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right\}$$

$$\pi = 12\sqrt{190}\ log\left\{(2\sqrt{2}+\sqrt{10})(3+\sqrt{10})\right\}$$

formulae correct to 9, 16 and 18 decimal places; and

$$\frac{1}{\pi} = 2\sqrt{2}\left\{\frac{1103}{99^2} + \frac{27493}{99^6}\cdot\frac{1}{2}\cdot\frac{1.3}{4^2} + \frac{53883}{99^10}\cdot\frac{1.3}{2.4}\cdot\frac{1.3.5.7}{4{\cdot}8^2} + \cdots\right\}$$

$$= 2\sqrt{2}\sum_{n=0}^{\infty}\frac{(1/4)_n(1/2)_n(3/4)_n}{(1)_n(1)_n(1)_n}(1103+26390n)(\frac{1}{99})^{4n+2},$$

which he asserted would be "*rapidly convergent*", for the first term itself gives the sum to 8 decimal places. In 1986, Borwein and Borwein [64], two computer scientists used a version of Ramanujan's formula to calculate π to

[1]It is the ancient Greeks who set-up the circle-squaring problem with the following two conditions [83]: the solution should use only a straightedge and compass, so that the proofs can be reduced to Euclid's theorems and the solution must use only a finite number of steps. If, in fact, $\pi r^2 = a^2$, where r is the radius of the circle and a is the length of the side of a square, then it follows that $\pi = (a/r)^2$, or that π is rational. In fact, π has been shown to be not only an *irrational number* but also, proved by Ferdinand von Lindermann, in 1882, to be a *transcendental* number. Furthermore, the phrase 'squaring the circle' in common usage suggests a project doomed to failure!

[2]Note that Ramanujan has used the equality sign for his approximations to π even though he states explicitly that the formulae are correct only to a certain number of decimal places.

17 million places and found that the formula converges on the exact value with far greater efficiency than any previous method[3]. This success proved that Ramanujan's insight was correct.

The next entries on this page in Ch. XVIII of his second note book are solutions of Diophantine equations:

$$\begin{aligned} X^3 + Y^3 &= U^2 \\ \text{and} \qquad X^3 + Y^3 + Z^3 &= U^3 \quad \text{(Euler's equation).} \end{aligned}$$

which are followed by numerical examples for these equations – 6 solutions for the first and 12 solutions for the Euler's equation and one which looks like but is not a solution of the Euler's equation. Characteristically, Ramanujan does not give the values of the parameters for the examples (ref. [VIII], part III, p.199 for the examples with the values of the parameters appended).

Ramanujan's solution of Euler's equation in rational numbers given here is:

$$\begin{aligned} X &= m^7 - 3m^4(1+p) + m(2+6p+3p^2), \\ Y &= 2m^6 - 3m^3(1+2p) + 1 + 3p + 3p^2, \\ Z &= m^6 - 1 - 3p - 3p^2, \\ U &= m^7 - 3m^4 p + m(3p^2 - 1), \end{aligned}$$

where m and p denote arbitrary numbers. He has also given another solution of this equation elsewhere (see Question 441 which follows) but neither of these two parameter solutions is the general solution[4] which contains three parameters, prompting Hardy [65] to comment that

> *Diophantine equations should have suited him, but he did comparatively little with them, and what he did was not his best.*

The second of the 12 examples given for the Euler equation on this page:

$$1^3 + 12^3 = 9^3 + 10^3 \quad (= \ 1729)$$

[3]David and Gregory Chudnovsky are two mathematicians who hold several world records for calculating the highest number of digits of π: first up to 450 million digits, then 1 billion and then 2 billion digits. The current record is over 51 billion digits (cf. For a brief history of π, [83]).

[4]The last of the examples $1^3 + 6^3 + 8^3 = 9^3$ does not belong to this class of solutions, since no values of m and p yield this solution to the Euler's equation. So, its presence reveals that perhaps, Ramanujan was aware that his solution was not the most general for Euler's equation.

corresponds to ($m = 2$, $p = 3$). This has been made famous by the following anecdote associated with it, due to Hardy [66]:

> *He could remember the idiosyncrasies of numbers in an almost uncanny way. It was Littlewood who said that every positive integer was one of Ramanujan's personal friends. I remember going to see him once when he was lying ill in Putney. I had ridden in taxi-cab number 1729, and remarked that the number seemed to me rather a dull one, and that I hoped that it was not an unfortunate omen. 'No', he replied, 'it is a very interesting number, it is the smallest number expressible as a sum of two cubes in two different ways'. I asked him, naturally, whether he could tell me the solution of the corresponding problem for fourth powers; and he replied, after a moment's thought, that he knew no obvious example, and supposed that the first such number must be very large.*

The simplest known solution of:

$$x^4 + y^4 = z^4 + t^4$$

is Euler's: $59^4 + 158^4 \ = \ 133^4 + 134^4 \ = \ 635318657.$

Euler gave a solution involving two parameters, but no 'general' solution is known.

The anecdote reveals Ramanujan's remarkable feeling for numbers and his sharp memory which made him recall one entry out of several thousands he had made in his notebook just like that and the fact that he had not recorded in his notebook the observation he made about 1729 which came from him only when Prof. Hardy made an innocuous statement regarding the taxi number.

Though the number 1729 itself finds no explicit mention in the notebooks, the Euler's equation and/or its solution: $1^3 + 12^3 = 9^3 + 10^3$ finds mention[5] in three Questions to the Journal of the I.M.S. ([III], p.331) and in two places in his notebooks ([VII], Vol. 2, p.266 and 387). These are indicated below:

[5] I am thankful to Mr. P.K. Srinivasan for drawing my attention to these and his publications [81].

• **Question 441** (V, 39)[6]:
Shew that

$$(6a^2-4ab+4b^2)^3 = (3a^2+5ab-5b^2)^3+(4a^2-4ab+6b^2)^3+(5a^2-5ab-3b^2)^3,$$

and find other quadratic expressions satisfying similar relations.
[*Solution* by S. Narayanan, VI, 226.]

• **Question 661** (VII, 119):
Solve in integers $x^3+y^3+z^3=u^6$,
and deduce the following:

$$\begin{array}{rclrcl} 6^3-5^3-3^3 &=& 2^6, & 8^3+6^3+1^3 &=& 3^6, \\ 12^3-10^3+1^3 &=& 3^6, & 46^3-37^3-3^3 &=& 6^6, \\ 174^3+133^3-45^3 &=& 14^6, & 1188^3-509^3-3^3 &=& 34^6. \end{array}$$

[*Solutions by* N.B. Mitra, XIII, 15 – 17. Additional solution and remarks by N.B. Mitra, XIV, 73 – 77.]

• **Question 681** (VII, 160):
Solve in integers $x^3+y^3+z^3=1$,
and deduce the following:

$$\begin{array}{rclrcl} 6^3+8^3 &=& 9^3-1, & 9^3+10^3 &=& 12^3+1, \\ 135^3+138^3 &=& 172^3-1, & 791^3+812^3 &=& 1010^3-1, \\ 11161^3+11468^3 &=& 14258^3+1, & 65601^3+67402^3 &=& 83802^3+1. \end{array}$$

[*Partial solution by* N.B. Mitra, XIII, 17. See also N.B. Mitra, XIV, 73–77 (76–77).]

• If p,q,r are quantities so taken that $p+3a^2=q+3ab=r+3b^2=(a+b)^2$ and m and n are any two quantities, then

$$n(mp+nq)^3+m(mq+nr)^3 = m(np+mq)^3+n(nq+mr)^3.$$

A particular case of the above theorem is:

$$(3a^2+5ab-5b^2)^3+(4a^2-4ab+6b^2)^3+(5a^2-5ab-3b^2)^3 = (6a^2-4ab+4b^2)^3.$$

([VIII], Vol.2, p.266)

[6] J. of the IMS, vol. V, p.39.

- If $\alpha^2 + \alpha\beta + \beta^2 = 3\lambda\gamma^2$, then

$$(\alpha + \lambda^2\gamma)^3 + (\lambda\beta + \gamma)^3 = (\lambda\alpha + \gamma)^3 + (\beta + \lambda^2\gamma)^3.$$

([VIII], Vol.2, p.387)

Note that in the above 4-parameter solution for the Euler's equation, it has been observed (cf. [95] (c), p.43) that for $\alpha = 3, \beta = 0, \lambda = 3$ and $\gamma = 1$, we get $12^3 + 1^3 = 10^3 + 9^3 = 1729$. The fact that Ramanujan could come out spontaneously with the statement that 1729 in the smallest number which could be expressed as the sum of two cubes in two different ways, something which he had not jotted down in his notebooks, proves that every integer is a 'personal friend' of Ramanujan as remarked by Prof. Littlewood.

Hardy's statement that two thirds of Srinivasan Ramanujan's work in India, (contained in the notebooks), consisted of rediscoveries, is refuted by Prof. Bruce C. Berndt, who states after his extensive studies that Hardy's estimate is too high. Berndt's main goal was: "*To prove each of Ramanujan's theorems*" and for known results refer to literature where proofs may be found. Tables of the contents of "Ramanujan's Notebooks", Parts I to V, edited by Prof. Berndt are given here in the following pages. The titles of the chapters gives the reader an idea about the nature and range of problems and the number of results Ramanujan had recorded in each chapter.

In the Introduction to Part V, Berndt states:

> *This volume, however, should not be regarded as the closing chapter on Ramanujan's notebooks. Instead, it is just the first milestone on our journey to understanding Ramanujan's ideas. ... It is our fervent wish that these volumes will serve as springboards for further investigations by mathematicians intrigued by Ramanujan's remarkable ideas. ... Ramanujan remarked that several of his series arose from alternative theories of elliptic functions ... The first of the three alternative theories is the most interesting and the most important, and we feel that a large body of work remains to be discovered here.*

Ramanujan's Note Books, Part I, Bruce C. Berndt ([VIII], 1985):

Ch.	Subject	# of Results
1.	Magic Squares	43
2.	Sums Related to the Harmonic series or the Inverse Tangent Function	68
3.	Combinatorial Analysis and Series Inversions	86
4.	Iterates of the Exponential Function and an Ingenius Formal Technique	50
5.	Eulerian Polynomials and Numbers, Bernoulli Numbers and the Riemann Zeta-Function	94
6.	Ramanujan's Theory of Divergent series	61
7.	Sums of Powers, Bernoulli Numbers and the Γ Function	110
8.	Analogues of the Gamma Function	108
9.	Infinite Series Identities, Transformations, and Evaluations	139
	Ramanujan's Quarterly Reports	

Ramanujan's Note Books, Part II, Bruce C. Berndt ([VIII], 1989):

Ch.	Subject	# of Results
10.	Hypergeometric Series I	116
11.	Hypergeometric Series II	103
12.	Continued Fractions	113
13.	Integrals and Asymptotic Expansions	92
14.	Infinite Series	87
15.	Asymptotic Expansions Modular Forms	94

Ramanujan's work on hypergeometric series, contained in chapters X and XI of his second notebook was edited by Hardy, as stated earlier. In chapter 4 of this book, we take a closer look at some of the beautiful work of Ramanujan on hypergeometric series.

Ramanujan's Notebooks, Part III, Bruce C. Berndt ([VIII], 1991):

Ch.	Subject	# of Results
16.	q-Series and Theta Functions	134
17.	Fundamental Properties of Elliptic Functions	162
18.	The Jacobian Elliptic Functions	135
19.	Modular Equations of Degrees 3, 5 and 7 and Associated Theta Function Identities	185
20.	Modular Equations of Higher & Composite Degrees	173
21.	Eisenstein series	45

The 100 Pages of Unorganized material after Chapter 21 of Srinivasa Ramanujan's second Notebook and the 33 Pages of his third Notebook were organized into chapters 22 to 31 by Berndt.

Ramanujan's Notebooks, Part IV, Bruce C. Berndt ([VIII], 1994):

Ch.	Subject	# of Results
22	Elementary results	47
23.	Number Theory	108
24.	Theory of Prime numbers	24
25.	Theta Function and Modular Equations	86
26.	Inversion Formulas for Lemniscate & other functions.	10
27.	Q series	9
28.	Integrals	63
29.	Special Functions	39
30.	Partial Fraction Expansions	15
31.	Elementary and miscellaneous analysis	36
	16 Chapters of first Notebook	54

In the concluding part V, Berndt examines the unorganized pages in all the three notebooks of Ramanujan on continued fractions, alternative theories of elliptic functions, class invariants and singular moduli, explicit values of theta-functions, modular equations, infinite series, approximations and asymptotic expansions, etc. Berndt points out that *very few claims in this volume pertain to Ramanujan's published papers and problems.*

Ramanujan's Notebooks, Part V, Bruce C. Berndt ([VIII], 1997)[7]:

Ch.	Subject	# of Results
32.	Continued Fractions	73
33.	Ramanujan's Theories of Elliptic Functions to Alternative Bases	62
34.	Class Invariants and Singular Moduli	196
35.	Values of Theta-Functions	24
36.	Modular Equations and Theta-Function Identities in Notebook 1	87
37.	Infinite Series	53
38.	Approximations and Asymptotic Expansions	46
39.	Miscellaneous Results in the First Notebook	24

Many theorems communicated by Ramanujan to Hardy in his Jan.16, Feb. 27, 1913 letters are found in Chapters.22 – 31. In all, $759+605+834+491+565 = 3254$ theorems have been studied in the parts I–V by Prof. Berndt. Hardy had estimated that the notebooks of Ramanujan contained approximately 3000 – 4000 statements of theorems and besides editing one chapter of Ramanujan's second Notebook, in 1923, he tried to get them edited. His intentions have at long last been fulfilled by the tireless efforts of the indefatigable Bruce C. Berndt. For the motivated student of Mathematics, this

[7]The author thanks Prof. Berndt for readily communicating the Preface and the Contents pages of Part V, by e-mail, on request.

five part publication will provide a plethora of results for further studies. As stated elsewhere [XV], it is no exaggeration to say that as long as people do mathematics, the work of Ramanujan and the stupendous effort of Prof. Bruce C. Berndt in editing the Ramanujan notebooks will be appreciated.

R.P. Agarwal's work on the notebooks

Prof. Ratan P. Agarwal is the founder of a school of ordinary and basic hypergeometric series in India. In recent times, he has consolidated a part of his work in a publication entitled: *Resonance of Ramanujan's Mathematics* [82]. This two volume work concerns some of the results of Srinivasa Ramanujan (1887 - 1920), the renowned Indian Mathematician, recorded in his notebooks, which continue to be a perennial source of inspiration for generations of Mathematicians.

Volume I has five chapters. Chapter 1 concerns certain summation and asymptotic formulae for generalized (ordinary and basic) hypergeometric series. Chapter 2 is on Rogers-Ramanujan identities. Chapters 3 - 5 deal with certain definite Integrals of interest to Ramanujan.

Prof. Agarwal opines, like Hardy, that *a major part of Ramanujan's systematic work on ordinary hypergeometric series has been in the nature of a rediscovery of the work already done by previous authors* – the ${}_7F_6(1)$ summation formula of Dougall (1907) and particular cases of this sum, such as, Pfaff (1797) - Saalschütz (1890) theorem for the ${}_3F_2(1)$ and the Gauss and Kummer summation theorems for the ${}_2F_1(1)$ and the ${}_2F_1(-1)$, respectively, and Dixon's summation theorem for a well-poised ${}_3F_2(1)$. These and Ramanujan's entries in his notebooks concerning the deep results on the asymptotic relations for a finite number of terms of a variety of ordinary hypergeometric series, which center around the behaviour of the logarithmic solutions of the hypergeometric differential equation are presented in Chapter 1.

Chapter 2 deals with the derivations of the celebrated Rogers–Ramanujan identities[8], their combinatorial interpretations, their generalizations and their

[8]Refer chapter 2 of this work for a discussion about these famous identities.

applications in Statistical Mechanics for the solution of the two-dimensional hard-hexagonal model (by R.J. Baxter). Several proofs of these identities and the interesting mathematical developments have been described very well in this cogently written chapter.

The next three chapters of Volume I deal with some of the integrals studied by Ramanujan. Prof. Agarwal presents definite integrals associated with Fourier transforms in the second Notebook and self-reciprocal functions given in his 'lost' Notebook in Chapter 3. He has shown how some of Ramanujan's results on integrals in his Notebook can be established with the help of other entries of his

> *that every entry given in the notebooks by him has a purpose, although sometimes a particular entry may look out of context at the first instance. ... We have advocated elsewhere also that it may be interesting many a time to try to prove 'unproven' entry of Ramanujan with the help of some other entry given by him earlier or later on.*

Chapter 4 is focused on Ramanujan's Master Theorem for the evaluation of definite integrals, viz.:

$$\int_0^\infty x^{n-1} F(x) dx = \Gamma(n)\phi(-n), \qquad \text{where} \quad F(x) = \sum_{k=0}^{\infty} \frac{\phi(k)(-x)^k}{k!},$$

in some neighbourhood of $x = 0$ and for n not necessarily a positive integer. The three Quarterly Reports submitted by Ramanujan to the Madras University, as a requirement stipulated by the University for awarding him the *First Research Scholarship in Mathematics*, contain various applications of this theorem to the *evaluation of a huge variety of integrals* and expansion formulae. The appropriate conditions of convergence for its validity have been provided by Hardy in his lectures contained in *Ramanujan* [IV]. The q-extension of the Master Theorem by Jackson (1951) and its extension to two variables by R.P. Agarwal (1974) are presented. The fundamental gamma and beta functions of mathematical analysis have been ingeniously extended by Ramanujan and these are related to the 'incomplete' gamma and beta functions (in which the upper limit of the integrals is finite and not ∞). The q-extensions of these works by several mathematicians including

Thomae (1879), F.H. Jackson (1904), R. Askey (1978), G.E. Andrews and R. Askey (1981), A. Verma and V.K. Jain (1992) are also included.

Chapter 5 is devoted to the study of integrals of the type:

$$\int_{-\infty}^{\infty} \frac{e^{at^2+bt}}{e^{ct}+d} dt$$

for particular values of a, b, c, d, evaluated by Ramanujan. Mordell (1933) classified these according to the values of the parameters and evaluated them in terms of Jacobi's theta and other related functions, using the method of complex contour integration. Professor Agarwal presents the general method of evaluation of different standard forms of this integral and relates them to those studied by Ramanujan, who used transform calculus to evaluate his integrals. Integrals associated with Riemann's zeta functions and elliptic modular relations are not discussed, as pointed out by the author himself. The chapter concludes with the mention of the example of a definite integral with fractional derivatives in the integrand stated in the third Quarterly Report. Ramanujan's formal deduction is based on his Master theorem and it agrees with the classical Liouville's definition of a fractional derivative of a function.

In Volume II, Prof. Agarwal presents a critical appraisal of Ramanujan's extensive and intriguing work on elliptic functions. Results pertaining to theta functions, partial theta functions, 'mock' theta functions (discovered by Ramanujan during the last year of his life on his return to India), as well as Lambert series and their relationship with elliptic functions, 'mock' theta functions and allied functions. While theta functions and partial theta functions are covered in chapter 1 of this volume and Lambert series and related functions are dealt with in chapter 5, the major part of this volume is concerned with the 'mock' theta function results of Ramanujan, presented in three extensive chapters (2, 3 and 4), contained in the 'Lost' Notebook, which was made available to the world in a facsimile edition, by Narosa, for the first time in the centenary year of Ramanujan's birth 1987 (though it was discovered in the estate of G.N. Watson by Prof. George E. Andrews of the Pennsylvania State University, in Spring 1976).

After giving the definitions of the four Jacobi theta functions and the 'false' theta functions (viz. theta functions with the 'wrong' signs for the terms) defined by L.J. Rogers, Prof. Agarwal introduces the Ramanujan notation for the theta function in terms of his symmetric function:

$$f(a,b) = \sum_{k=-\infty}^{\infty} (ab)^{k(k-1)/2}(a^k + b^k), \quad |ab| < 1.$$

In terms of this symmetric function, Ramanujan defines:

$$f(qe^{2iz}, qe^{-2iz}) = \theta_3(z,q), \text{ where } q = e^{i\pi\tau} \text{ and } |q| < 1.$$

Several general properties of the function $f(a,b)$ are then presented. Jacobi's triple product identity and other such identities due to D. Hickerson (1988) and G.E. Andrews and D. Hickerson (1991) are derived explicitly, due to their importance to simplify calculations in later chapters on 'mock' theta functions. A number of theta function expansions in the 'lost' notebook pertain to 'partial' theta functions. Some of these are stated and G.E. Andrews (1981) derived a general basic hypergeometric identity and showed the results of Ramanujan as special cases of it.

Ramanujan defined four third order mock theta functions and to this set three more were added by Watson. Watson used the transformations of basic hypergeometric series for obtaining new definitions for all the seven functions. Agarwal himself has contributed significantly to the theory of 'mock' theta functions by showing that most of the general identities of Andrews (referred above) belong to a very general class of basic hypergeometric transformations. His results are presented and those of Andrews deduced from them. Furthermore, Agarwal defines the third order mock theta functions in terms of a ${}_2\Phi_1$ basic hypergeometric series and deduces many of the mock theta function properties from the well-known properties of the ${}_2\Phi_1$. Ramanujan gave ten mock theta functions of order five, in two groups of five each, and three functions of order seven. He asserted that the members of each of the two groups of mock theta functions of order five are related amongst those belonging to the same group only, while the three mock theta functions of order seven are not related to each other. The works of Andrews, Agarwal and M. Gupta on the mock theta functions of order five are presented in detail. A. Gupta and Agarwal showed that mock theta functions of order five and seven are defined through basic hypergeometric series of type ${}_3\Phi_2$ and ${}_4\Phi_3$. They have also made attempts to develop the theory of the above types of basic hypergeometric series to obtain new transformations and definitions for certain mock theta functions of orders five and seven. It is pointed out that the absence of a general transformation theory for the ${}_3\Phi_2(a, b, c; e, f; z)$ series [except when the argument z is q (the base parameter itself) or ef/abc], is the main reason why there exists no general transformation theory for mock theta functions of order five.

Observations of Agarwal on the relationship between the 'mock' theta functions and the basic hypergeometric series lead him to define:

> *the order of a mock theta function as* $(2r+1)$ *if it is expressible in terms of a* ${}_{r+1}\Phi_r$ *series, on a single base* q^k, $k \leq r+1$. *There may be in the definition of the mock theta function an additive term with* ${}_{r+1}\Phi_r$ *consisting of* θ*-products, which do not affect the order.*

While this is an intersting observation by itself, it is simplistic, to say the least. For, mystery still surrounds the incomplete work of Ramanujan in this area contributing to the belief that Ramanujan was working on a general theory of mock theta functions, whose preliminary results are the ones in the 'lost' notebook of his and that he was snatched away by fate before he could reveal his grand scheme!

This two volume work of Prof. Agarwal is a deep study of some of the work of Ramanujan on ordinary and basic hypergeometric series, on definite integrals and theta, partial theta and mock theta functions. It is very original in parts and has a minimum overlap with the comprehensive five part work on the notebooks of Ramanujan by Bruce C. Berndt[9]. This work presents, in as coherent a way as is possible, the work of Agarwal and his school of students and coworkers. It is a valuable supplement for the research scholar and for mathematicians and presents the author's understanding of the 'how' and 'why' of what Ramanujan did.

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[9] The author's reference to these Parts I to V as the Berndt Notebooks is unfortunate, especially in view of the significance of the Ramanujan notebooks and the 'Lost' notebook, which are the source materials for both Berndt and Agarwal.

Dyson on Ramanujan's notebooks

Freeman J. Dyson, the renowned Physicist ([XVI], p.25), concludes his article entitled: *A Walk through Ramanujan's Garden*, with the following advise:

> *In conclusion, I would like to urge all of you who are working in the many fields of mathematics which have been enriched by Ramanujan's ideas to go back to the source the collected papers and the notebooks. ... The notebooks ... are now appearing in a splendidly annotated version edited by Bruce Berndt [VIII]. The "lost" notebook is now accessible to us through the devoted labors of George Andrews ([53], [56]). When I started my walk through Ramanujan's garden 47 years ago, only the collected papers were available. A year after I chose Hardy and Wright's "Theory of Numbers" (Oxford, Clarendon Press, 1938) as a school prize, I won another prize. For the second prize I chose Ramanujan's collected papers. The collected papers have traveled with me from England to America and are still as fresh to-day as they were in 1940. Whenever I am angry or depressed, I pull down the collected papers from the shelf and take a quiet stroll in Ramanujan's garden. I recommend this therapy to all of you who suffer from headaches or jangled nerves. And Ramanujan's papers are not only a good therapy for headaches. They also are full of beautiful ideas which may help you to do more interesting mathematics.*

Certainly several of us will not find this therapy useful. Since Dyson, Erdös and Selberg were among the very best in their chosen fields, they were greatly benefited by an early introduction to the Collected Papers. The rest can derive the benefit of the extensive study of Ramanujan's work by Hardy [IV] and Berndt [VIII] to begin appreciating the prolific, original work of the mathematical genius in his published work and in his notebooks.