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Surjectivity of power maps of real algebraic groups

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Abstract

In this paper we study the surjectivity of the power maps $g \mapsto g^n$ for real points of algebraic groups defined over reals. The results are also applied to study the exponentiality of such groups.

Keywords: Real algebraic groups; Power maps; Surjectivity; Exponentiality

1. Introduction

Let G be a real Lie group with Lie algebra $L(G)$ and let $\exp : L(G) \to G$ be the associated exponential map. The Lie group G is said to be *exponential* if $G = \exp(L(G))$. For an integer n, let P_n denote the *n-th power map* defined by $P_n(g) = g^n$, $g \in G$. It is then immediate that $P_n : G \to G$ is surjective if and only if $P_p : G \to G$ is surjective for all prime divisors p of n. A well-known result, proved independently by K. Hofmann, J. Lawson and M. McCrudden (see [19,21] and Theorem 4.16), says that a real Lie group is exponential if and only if all its n-th power maps are surjective. A considerable amount of work has been done on the so-called "exponentiality problem" for real groups, the main theme of which is finding criteria to decide which real Lie groups are exponential; the reader is referred to the survey article [14]. The exponentiality problem of real Lie groups is fairly well understood in the case of solvable groups and (semi)simple real Lie groups through the work of D. Djoković and N. Thang in [16] and M. Wüstner in [27,28]. But in the case of mixed groups, that is, groups with nontrivial Levi factor and solvradical, the general problem remains largely open. In the latter case when the group admits a compact Levi factor advances were made in [11] and [22] and for a general complex

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algebraic group a satisfactory solution follows from [7]. However, many Lie groups fail to be exponential and invoking the above result of Hofmann, Lawson and McCrudden, it is obvious that Lie groups fail to be exponential only in the presence of non-surjective n-th power maps. In view of this, it is natural to ask for a characterization of real Lie groups for which an individual n-th power map is surjective. On the other hand an answer to this question on the power maps on real Lie groups will have immediate implications on the exponentiality problem of such groups. The purpose of this paper is to address such questions on the Lie groups which arise as the group of real points $G(\mathbb{R})$ of a complex algebraic group G defined over $\mathbb{R}$.

To put the present work in proper perspective we briefly mention the work that has been done on the n-th power maps and its ramifications to the exponentiality problem. In [6] we obtained a necessary and sufficient condition for the surjectivity of the n-th power maps for connected solvable real Lie groups in terms of Cartan subgroups. This result has many applications which include strengthening of Dixmier's characterization of solvable simply connected real Lie groups which are exponential. In [7] we have given a characterization for the surjectivity of the n-th power maps for general connected algebraic groups over algebraically closed fields of characteristic zero in terms of a maximal torus and its weights, which in turn yields a complete solution of the exponentiality problem, for this class of groups. As further applications, in [7] we have explicitly determined, for all simple algebraic groups over algebraically closed fields of characteristic zero, the set of integers n for which the n-th power map is surjective. The author in [8] and R. Steinberg in [26], independently extended some of the results of [7] which leads to a complete classification of exponents n for which P_n is surjective for semisimple groups over algebraically closed fields of arbitrary characteristic. More recently, in [9] we have solved the question of surjectivity of n-th power maps of p-adic algebraic groups and studied its relation with the exponentiality of such groups, and in [10] we have obtained results on these questions for real Lie groups and algebraic groups, which are not necessarily (Zariski) connected.

We now describe the main results of this paper. The reader is referred to the next section for the relevant notations and definitions.

Theorem 1.1. *Let G be a complex algebraic group defined over $\mathbb{R}$ and H be an $\mathbb{R}$-algebraic subgroup of maximal rank. Suppose that $P_n : G \to G$ is surjective. Then the following hold.*

(1) *Let n be an odd integer, H be Zariski connected and T be a maximal torus of H which is defined over $\mathbb{R}$. Then $P_n : Z_{T(\mathbb{R})}(X) \to Z_{T(\mathbb{R})}(X)$ is surjective for all nilpotent $X \in L(H(\mathbb{R}))$.*
(2) *Let n be an odd integer. Then $P_n : H(\mathbb{R})^* \to H(\mathbb{R})^*$ is surjective. Moreover, if n is coprime to the order* $\mathrm{Ord}(H(\mathbb{R})/H(\mathbb{R})^*)$ *then $P_n : H(\mathbb{R}) \to H(\mathbb{R})$ is surjective. In particular, if H is Zariski connected then $P_n : H(\mathbb{R}) \to H(\mathbb{R})$ is surjective.*
(3) *In* (2) *above, if H further admits an $\mathbb{R}$-anisotropic maximal torus then we do not need to restrict n to be odd, that is, $P_n : H(\mathbb{R})^* \to H(\mathbb{R})^*$ is surjective. Moreover, if n is coprime to the order* $\mathrm{Ord}(H(\mathbb{R})/H(\mathbb{R})^*)$ *then $P_n : H(\mathbb{R}) \to H(\mathbb{R})$ is surjective.*

If H is Zariski connected and n is odd then $P_n : Z(H(\mathbb{R})) \to Z(H(\mathbb{R}))$ is surjective.

In Sections 4 and 5 of [7] methods were developed to determine, for complex algebraic groups G, the exact set of integers n for which $P_n : G \to G$ is surjective. Equipped with this, Theorem 1.1 becomes an effective tool in finding integers n for which P_n is surjective or establishing exponentiality for Lie groups that appear as the group of real points of maximal rank subgroups of complex algebraic groups. This is illustrated in Theorem 1.3, Corollary 1.4 and Corollary 1.5.

Next, as an immediate application of Theorem 1.1 and some results of [7] we get the following corollary.

Corollary 1.2. *Let G be a complex algebraic group defined over $\mathbb{R}$. Then there exists an integer m_G such that $P_n : G(\mathbb{R}) \to G(\mathbb{R})$ is surjective for all n coprime to m_G.*

In other words, Corollary 1.2 says that if G is as above then $P_p : G(\mathbb{R}) \to G(\mathbb{R})$ is surjective for all but finitely many primes p. Considering a connected linear solvable Lie group and the simply connected cover $\widetilde{SL_2(\mathbb{R})}$ of $SL_2(\mathbb{R})$ in Example 4.10 and in Example 4.11, respectively, we show that this no longer holds for general connected real Lie groups.

In the next result we use Theorem 1.1 and Theorems A, C and Corollary D of [7] to get a more precise form of Corollary 1.2 for the class of semisimple groups over $\mathbb{R}$.

Theorem 1.3. *Let G be a complex algebraic group defined over $\mathbb{R}$ and H be a Zariski connected $\mathbb{R}$-algebraic subgroup of G of maximal rank.*

(1) *If G is semisimple and of classical type and n is odd then $P_n : H(\mathbb{R}) \to H(\mathbb{R})$ is surjective if n is coprime to* $\mathrm{Ord}(Z(H(\mathbb{R}))/Z(H(\mathbb{R}))^*)$. *In particular, if G is $\mathbb{R}$-simple of type B_l, C_l or D_l then $P_n : H(\mathbb{R}) \to H(\mathbb{R})$ is surjective if n is an odd integer.*
(2) *If G is $\mathbb{R}$-simple of type E_6, E_7, F_4 or G_2 then $P_n : H(\mathbb{R}) \to H(\mathbb{R})$ is surjective if n is coprime to* 6.
(3) *If G is $\mathbb{R}$-simple of type E_8 then $P_n : H(\mathbb{R}) \to H(\mathbb{R})$ is surjective if n is coprime to* 30.

Thus, if G is a semisimple group of classical type and n is coprime to $2 \cdot \mathrm{Ord}(Z(G(\mathbb{R})))$ *then $P_n : G(\mathbb{R}) \to G(\mathbb{R})$ is surjective. More generally, if G is any semisimple group over $\mathbb{R}$ then $P_n : G(\mathbb{R}) \to G(\mathbb{R})$ is surjective if n is coprime to* $30 \cdot \mathrm{Ord}(Z(G(\mathbb{R})))$.

We remark that the above result extends the “if” part of Theorem C [7].

We now state our next result on the exponentiality of groups $G(\mathbb{R})$.

Corollary 1.4. *Let G be a Zariski connected complex algebraic group defined over $\mathbb{R}$ and H be a Zariski connected $\mathbb{R}$-algebraic subgroup of G of maximal rank.*

(1) *Suppose for all odd integers n the map $P_n : G \to G$ is surjective. Then $H(\mathbb{R})$ is exponential if and only if $P_2 : H(\mathbb{R}) \to H(\mathbb{R})$ is surjective.*
(2) *Suppose G is exponential and H admits an $\mathbb{R}$-anisotropic maximal torus. Then the group $H(\mathbb{R})^*$ is exponential.*

Consequently, if G is semisimple and of classical type and $H(\mathbb{R})$ is evenly centered then $H(\mathbb{R})$ is exponential if and only if $P_2 : H(\mathbb{R}) \to H(\mathbb{R})$ is surjective.

We now record Corollary 1.5 which is a straightforward application of Theorem 1.3 and Corollary 1.4. In dealing with the exponentiality problem for minimal parabolic subgroups of $SL_n(\mathbb{H})$, S.G. Dani and M. McCrudden arrive at Corollary 1.2 [11], concerning the relation between square roots and exponentiality of individual elements in minimal parabolic subgroups of $GL_n(\mathbb{H})$. It follows readily from Corollary 1.2 of [11] that, for any n, a minimal parabolic subgroup P in $SL_n(\mathbb{H})$ is exponential if and only if $P_2 : P \to P$ is surjective. We generalize this observation in (1) of Corollary 1.5 to any maximal rank subgroup of $SL_n(\mathbb{H})$. In (2) of Corollary 1.5, we obtain a new class of exponential subgroups of $U(p,q)$ and $PSU(p,q)$ and as an outcome, we get a new proof of the fact that $U(p,q)$ is an exponential group, which was proved earlier in [12] using the description of conjugacy classes of $U(p,q)$.

Corollary 1.5. *Let G be a complex algebraic group over $\mathbb{R}$ and H be a Zariski connected maximal rank subgroup defined over $\mathbb{R}$.*

(1) *If $G = SL_{2m}(\mathbb{C})$ equipped with the $\mathbb{R}$-structure such that $G(\mathbb{R}) = SL_m(\mathbb{H})$ and $H(\mathbb{R})$ is evenly centered then $P_n : H(\mathbb{R}) \to H(\mathbb{R})$ is surjective if n is odd. In particular, if P is any $\mathbb{R}$-parabolic subgroup of G then $P_n : P(\mathbb{R}) \to P(\mathbb{R})$ is surjective if n is odd. Thus $P(\mathbb{R})$ is exponential if and only if $P_2 : P(\mathbb{R}) \to P(\mathbb{R})$ is surjective.*
(2) *Let p, q be two integers such that $p + q = m$. If $G = GL_m(\mathbb{C})$ (resp. $G = PSL_m(\mathbb{C})$) is equipped with the $\mathbb{R}$-structure such that $G(\mathbb{R}) = U(p,q)$ (resp. $G(\mathbb{R})^* = PSU(p,q)$) then $P_n : H(\mathbb{R})^* \to H(\mathbb{R})^*$ is surjective if n is odd. Thus $H(\mathbb{R})^*$ is exponential if and only if $P_2 : H(\mathbb{R})^* \to H(\mathbb{R})^*$ is surjective. Let D denote a maximal compact torus* (*that is, maximal compact, connected, abelian subgroup*) *of $U(p,q)$* (*resp. of $PSU(p,q)$*)*. If $H(\mathbb{R})$ further contains a $G(\mathbb{R})$-conjugate of D then $H(\mathbb{R})^*$ is exponential.*

The next result gives sufficient conditions for the surjectivity of the n-th power maps. For relevant notations used in this result see Section 5 in addition to Section 2. Let G be a Zariski connected algebraic group defined over $\mathbb{R}$ and H be a Levi subgroup over $\mathbb{R}$. Let $\mathcal{A}_H$ be a set of all mutually nonequivalent admissible subsets of a set of simple roots $\Delta_{\mathbb{R}}^H$ in the $\mathbb{R}$-root system with respect to a fixed maximal $\mathbb{R}$-split torus S^H of H (see Definition 5.8). For each θ in $\mathcal{A}_H$ one gets the standard $\mathbb{R}$-parabolic subgroup P_θ^H of H which in turn yields the standard $\mathbb{R}$-parabolic subgroup $P_\theta = P_\theta^H R_u G$ of G. Further, for each θ in $\mathcal{A}_H$ we fix a maximal $\mathbb{R}$-torus T_θ of H such that $S_\theta^H \stackrel{\text{def}}{=} (\bigcap_{\alpha\in\theta} \operatorname{Ker}\alpha)^0$ is the $\mathbb{R}$-split part of T_θ.

Theorem 1.6. *Let G be a Zariski connected complex algebraic group defined over $\mathbb{R}$. Let n be an integer.*

(1) *Let H be a Levi subgroup of G which is defined over $\mathbb{R}$. Suppose that for every $\theta \in \mathcal{A}_H$ and for every $X \in L(R_u P_\theta(\mathbb{R}))$ the map $P_n : Z_{T_\theta(\mathbb{R})}(X) \to Z_{T_\theta(\mathbb{R})}(X)$ is surjective then $P_n : G(\mathbb{R}) \to G(\mathbb{R})$ is surjective.*
(2) *Let $\{T_1, \ldots, T_k\}$ be a set of representatives of the finitely many $G(\mathbb{R})$-conjugacy classes of maximal tori of G defined over $\mathbb{R}$. Suppose, for each i the map $P_n : Z_{T_i(\mathbb{R})}(X) \to Z_{T_i(\mathbb{R})}(X)$ is surjective, for all nilpotent elements $X \in L(G(\mathbb{R}))$ then $P_n : G(\mathbb{R}) \to G(\mathbb{R})$ is surjective.*
(3) *Let T be a maximal torus of G which is defined over $\mathbb{R}$ and maximally $\mathbb{R}$-anisotropic in G and let T_{an} be the anisotropic part of T. Suppose the map $P_n : Z_{T_{an}(\mathbb{R})}(X) \to Z_{T_{an}(\mathbb{R})}(X)$ is surjective for all positive elements $X \in L(G(\mathbb{R}))$ then $P_n : G(\mathbb{R})^* \to G(\mathbb{R})^*$ is surjective.*

The part (2) in the above theorem establishes one side of the power map analogue of a conjecture posed in Problem 5.6 [14]. See also Remark 5.12 on the sufficiency of the conditions in the above theorem.

In the next few results we deal with the interrelation between surjectivity of power maps on groups and their minimal parabolic subgroups. In response to Problem 5.7 of [14], regarding the simultaneous exponentiality of Lie groups and their minimal parabolic subgroups, S.G. Dani and M. McCrudden in [11] and D. Djoković in [13] independently proved that the minimal parabolic subgroups of $SL_n(\mathbb{H})$ are not exponential if $n \geqslant 8$. Further, in [11] it is shown that minimal parabolic subgroups in $SL_n(\mathbb{H})$, $n \leqslant 4$, are exponential, while in [13], among other results, it is proved that minimal parabolic subgroups in $U(p,q)$ are exponential. Recall that both $SL_n(\mathbb{H})$ and $U(p,q)$ are exponential groups. Thus, any speculation on the dependence of exponentiality of Lie groups and their minimal parabolic subgroups, seems to be false in general. However, Theorems 1.1 and 1.3 support the fact that the set of integers n, for which P_n is surjective for both the group and its minimal parabolic subgroup, could be large. Hence, fixing an integer n, it is natural to find the class of groups, for which surjectivity of P_n on the ambient Lie group is equivalent to that on its minimal parabolic subgroups. For any integer n, in [7] it is proved that this class contains all connected complex algebraic groups. In the next theorem we show, in particular, that if n is odd then any semisimple algebraic group over $\mathbb{R}$ of real rank one also belongs to the above class.

Theorem 1.7. *Let G be a connected semisimple algebraic group over $\mathbb{R}$ which is a direct product (as $\mathbb{R}$-algebraic groups) of either semisimple groups of $\mathbb{R}$-rank one or $\mathbb{R}$ algebraic groups which are evenly centered and of classical type. Let n be an odd integer and P be a minimal $\mathbb{R}$-parabolic subgroup of G. Then $P_n : G(\mathbb{R}) \to G(\mathbb{R})$ is surjective if and only if $P_n : P(\mathbb{R}) \to P(\mathbb{R})$ is surjective.*

Our next Theorem 1.9 relates the surjectivity of the n-th power maps and the existence of P_n-regular n-th roots of certain semisimple elements. This finally yields a necessary condition for the surjectivity of power maps for $\mathbb{R}$-quasisplit groups. The key fact, required in the proof of Theorem 1.9, which is of independent interest, is Theorem 1.8 which ensures the existence of a semiregular unipotent element in $H(\mathbb{R})$ for any $\mathbb{R}$-quasisplit group H. It was proved by L. Rothschild in Proposition 5.1 [23] that an $\mathbb{R}$-quasisplit reductive Lie algebra $\mathfrak{h}$ in fact admits a regular nilpotent element in $\mathfrak{h}(\mathbb{R})$. It is also shown in Proposition 3.1 [7] that a semiregular unipotent element exists in a general algebraic group over an algebraically closed field of characteristic zero. Thus Theorem 1.8 may be regarded as a generalization of the above result of Rothschild to any Zariski connected $\mathbb{R}$-quasisplit group which is not necessarily reductive. On the other hand, Theorem 1.8 extends Proposition 3.1 of [7].

Theorem 1.8. *Let G be any Zariski connected $\mathbb{R}$-quasisplit complex algebraic group and P be a minimal $\mathbb{R}$-parabolic subgroup of G. Then there is a unipotent element $u \in R_u P(\mathbb{R})$ which is semiregular in G.*

We now apply Theorem 1.8 to obtain Theorem 1.9.

Theorem 1.9. *Let G be a Zariski connected complex algebraic group over $\mathbb{R}$. Suppose that $P_n : G(\mathbb{R}) \to G(\mathbb{R})$ is surjective.*

(1) *Let s be a semisimple element in $G(\mathbb{R})$ such that $Z_G(s)^0$ is an $\mathbb{R}$-quasisplit algebraic group. Then there is $r \in G(\mathbb{R})$ such that $s = r^n$ and r is P_n-regular in G.*
(2) *In addition, if G is assumed to be $\mathbb{R}$-quasisplit and if P is a minimal $\mathbb{R}$-parabolic subgroup of G then $P_n : P(\mathbb{R}) \to P(\mathbb{R})$ and $P_n : Z(G(\mathbb{R})) \to Z(G(\mathbb{R}))$ are both surjective.*

The following corollary is an immediate application (and hence the proof is omitted) of (1) of Theorem 1.3 and (2) of Theorem 1.9.

Corollary 1.10. *Let G be a Zariski connected $\mathbb{R}$-split semisimple group of classical type and let S be a maximal $\mathbb{R}$-split torus in G. Let H be a Zariski connected $\mathbb{R}$-algebraic subgroup of G containing S. Suppose $Z(H(\mathbb{R}))$ is not a connected Lie group. Then $P_n : H(\mathbb{R}) \to H(\mathbb{R})$ is surjective if and only if n is odd.*

2. Notations and background

In this section we fix most of the notations, definitions and recall some standard facts, which will be used throughout this paper. A few specialized notations and definitions are mentioned as and when they occur later. The reader is referred to [2,5,24] for generalities in the theory of algebraic groups. Although many facts stated here hold for algebraic groups over arbitrary fields, for our purpose we will recall them in the setting of algebraic groups defined over reals.

The *center* of an abstract group Γ is denoted by $Z(\Gamma)$ and the *order* of Γ is denoted by $\mathrm{Ord}(\Gamma)$. If G is a Lie group or an algebraic group we denote the *Lie algebra of* G by $L(G)$. For an integer n, let P_n denote the *n-th power map* defined by $P_n(g) = g^n$, $g \in G$. For a subgroup H of G and a subset S of G, $Z_H(S)$ will denote the subgroup consisting of all elements of H which commute with every element of S. Similarly, for $X \in L(G)$ and H a closed subgroup of G, $Z_H(X)$ denotes the closed subgroup $\{h \in H \mid \mathrm{Ad}(h)X = X\}$. The *Zariski connected component* of an algebraic group G, containing the identity element, is denoted by G^0, while for a real Lie group H, we denote the *connected component* (in the real topology) by H^*, to avoid confusion with the Zariski connected component of algebraic groups. A Lie group is said to be *evenly centered* if either the center is connected or the number of connected components of the center is a power of two. Observe that center-free adjoint groups and real points of the classical groups of type B_l, C_l and D_l are examples of evenly centered groups. An element g in a real Lie group A is said to be *P_n-regular in A* if the linear transformation $\mathrm{Ad}(g) : L(A) \to L(A)$ does not have a nontrivial n-th root of unity in $\mathbb{C}$ as an eigenvalue.

Let now G be an algebraic group defined over $\mathbb{R}$. The real points of G (resp. of $L(G)$) is denoted by $G(\mathbb{R})$ (resp. $L(G)(\mathbb{R})$). For any element $x \in G(\mathbb{R})$ (resp. $X \in L(G(\mathbb{R}))$) the semisimple and the unipotent (resp. nilpotent) Jordan components of x (resp. X) will be denoted by x_s (resp. X_s) and x_u (resp. X_n) respectively; then we have $x = x_s x_u = x_u x_s$ and $x_s, x_u \in G(\mathbb{R})$ (resp. $X = X_s + X_n$ and $X_s, X_n \in L(G(\mathbb{R}))$). The Zariski closure of the group generated by any nontrivial unipotent element is connected and one-dimensional; moreover if the nontrivial unipotent element lies in $G(\mathbb{R})$ then the Zariski closure of the group generated by this element will also be defined over $\mathbb{R}$ and the real points of the group will be isomorphic to the group of additive reals. The set of unipotent (resp. nilpotent) elements of $G(\mathbb{R})$ (resp. $L(G(\mathbb{R}))$) is denoted by $\mathcal{U}_{(G(\mathbb{R}))}$ (resp. $\mathcal{N}_{L(G(\mathbb{R}))}$). We further recall a more refined version of Jordan decomposition, called the *complete Jordan decomposition*, available in the group $G(\mathbb{R})$ and in the Lie algebra $L(G(\mathbb{R}))$. The reader is referred to Proposition 2.4 [3] and Theorem 7.2, p. 431 [17] for details. An element $e \in G(\mathbb{R})$ (resp. $E \in L(G(\mathbb{R}))$) is said to be *compact or elliptic* if e (resp. $\exp(E)$) lies in a compact subgroup of $G(\mathbb{R})$. Observe that a compact element is necessarily semisimple. A semisimple element $h \in G(\mathbb{R})$ (resp. $H \in L(G(\mathbb{R}))$) is said to be *hyperbolic* if $h \in S(\mathbb{R})^*$ (resp. $H \in L(S(\mathbb{R})^*)$) for some $\mathbb{R}$-split torus S of G. For a semisimple element $s \in G(\mathbb{R})$ (resp. $S \in L(G(\mathbb{R}))$), there is a unique pair of elements s_e (resp. S_e) and s_h (resp. S_h) in $G(\mathbb{R})$ (resp. in $L(G(\mathbb{R}))$) such that s_e (resp. S_e) is compact, s_h (resp. S_h) is hyperbolic and $s = s_e s_h = s_h s_e$ (resp.

$S = S_e + S_h$). For $x \in G(\mathbb{R})$ (resp. $X \in L(G(\mathbb{R}))$), the element $(x_s)_e$ (resp. $(X_s)_e$) will be denoted by x_e (resp. X_e). Similarly the element $(x_s)_h$ (resp. $(X_s)_h$) will be denoted by x_h (resp. X_h). The *positive* part of x (resp. X), which is defined to be $x_h x_u$ (resp. $X_h + X_n$), is denoted by x_p (resp. X_p). An element $y \in G(\mathbb{R})$ (resp. $Y \in L(G(\mathbb{R}))$) is said to be *positive* if $y = y_h y_u$ (resp. $Y = Y_h + Y_n$). Thus we have the *complete Jordan decomposition* in $G(\mathbb{R})$ (resp. $L(G(\mathbb{R}))$) as follows. For all $z \in G(\mathbb{R})$ (resp. $Z \in L(G(\mathbb{R}))$), there is a unique mutually commuting triplet $z_e, z_h, z_u \in G(\mathbb{R})$ (resp. $Z_e, Z_h, Z_n \in L(G(\mathbb{R}))$) such that z_e (resp. Z_e) is compact, z_h (resp. Z_h) is hyperbolic, z_u is unipotent (resp. Z_n is nilpotent) and $z = z_e z_h z_u$ (resp. $Z = Z_e + Z_h + Z_n$). Moreover, written differently, for all $z \in G(\mathbb{R})$ (resp. $Z \in L(G(\mathbb{R}))$), there are commuting elements $z_e, z_p \in G(\mathbb{R})$ (resp. $Z_e, Z_p \in L(G(\mathbb{R}))$) such that z_e (resp. Z_e) is compact, z_p (resp. Z_p) is positive and that $z = z_e z_p$ (resp. $Z = Z_e + Z_p$). We will denote the set of positive elements in $G(\mathbb{R})$ (resp. $L(G(\mathbb{R}))$) by $\mathcal{P}_{G(\mathbb{R})}$ (resp. $\mathcal{P}_{L(G(\mathbb{R}))}$).

Let G be an algebraic group. The maximal, Zariski connected, Zariski closed, normal, unipotent subgroup of G is called the *unipotent radical* of G and it is denoted by $R_u G$. The *rank* of G is the dimension of any of its maximal tori. An algebraic subgroup H of G is said to be of *maximal rank* if rank of H is the same as the rank of G. A maximal, Zariski connected, solvable algebraic subgroup of G is called a *Borel subgroup* while an algebraic subgroup P of G containing a Borel subgroup is called a *parabolic subgroup*. In the case when G is defined over $\mathbb{R}$ and the parabolic subgroup P is an $\mathbb{R}$-algebraic subgroup of G then P is called a $\mathbb{R}$*-parabolic subgroup*.

Let now G be an algebraic group defined over $\mathbb{R}$. Recall that a torus T is said to be $\mathbb{R}$*-split* if T is isomorphic, as $\mathbb{R}$-algebraic groups, to a product of copies of $\mathbb{C}^*$. The $\mathbb{R}$*-rank* of G is the dimension of any of its maximal $\mathbb{R}$*-split* tori. If $\mathbb{R}$-rank of G is positive then G is called $\mathbb{R}$*-isotropic*; otherwise it is called $\mathbb{R}$*-anisotropic*. Let $\overline{T}$ be a torus defined over $\mathbb{R}$. The maximal $\mathbb{R}$-split (resp. $\mathbb{R}$-anisotropic) subtorus of $\overline{T}$ is denoted by $\overline{T}_{sp}$ (resp. $\overline{T}_{an}$). A maximal $\mathbb{R}$-torus T of G is said to be *maximally* $\mathbb{R}$*-anisotropic* if T_{an} has the maximum possible dimension, that is, $\dim T_{an} \geqslant \dim T'_{an}$ for any maximal torus T' defined over $\mathbb{R}$.

Let now G be reductive over $\mathbb{R}$ and S be a maximal $\mathbb{R}$-split torus of G. The set of $\mathbb{R}$*-roots* with respect to S and a set of *simple roots* thereof are denoted by $\Phi_{\mathbb{R}}$ and $\Delta_{\mathbb{R}}$, respectively. Then for every subset θ of $\Delta_{\mathbb{R}}$ one gets a natural $\mathbb{R}$-parabolic subgroup P_θ, called the *standard* $\mathbb{R}$*-parabolic subgroup corresponding to* θ (see [2,5,24] for details). Moreover, $P_\theta = Z_G(S_\theta) R_u P_\theta$ where $S_\theta = (\bigcap_{\alpha \in \theta} \operatorname{Ker} \alpha)^0$.

We now assume that G is a Zariski connected algebraic group, defined over $\mathbb{R}$, but G is not necessarily reductive. Let H be a (reductive) Levi subgroup over $\mathbb{R}$ such that G admits the *Levi decomposition*, $G = H R_u G$. Let S^H be a maximal $\mathbb{R}$-split torus of H, $\Delta^H_{\mathbb{R}}$ be a set of simple roots in the $\mathbb{R}$-root system of H with respect to S^H. Corresponding to the given Levi subgroup H of G and a subset ϕ of $\Delta_{\mathbb{R}}$, we define the *standard* $\mathbb{R}$*-parabolic subgroup of* G by $P_\phi = P^H_\phi R_u G$ where, P^H_ϕ is the standard $\mathbb{R}$-parabolic subgroup of the reductive group H corresponding to $\phi \subset \Delta^H_{\mathbb{R}}$, as defined above.

Let G be a complex algebraic group and σ be a generator of the Galois group of $\mathbb{C}$ over $\mathbb{R}$. Let G^σ denote the algebraic group obtained from conjugating G by σ. Then the product $G \times G^\sigma$ naturally acquires an $\mathbb{R}$-structure. This algebraic group, defined over $\mathbb{R}$, is called the *Weil restriction* of G and is denoted by $\mathcal{R}_{\mathbb{C}/\mathbb{R}} G$ (see Section 12.4, pp. 220–222 [24] for more generalities).

Recall that a non-abelian Zariski connected complex algebraic group (resp. over $\mathbb{R}$) is said to be *absolutely simple* (resp. $\mathbb{R}$*-simple*) if it does not admit any Zariski closed normal connected subgroup (resp. defined over $\mathbb{R}$) of positive dimension. Recall that an $\mathbb{R}$-simple group G is either absolutely simple or of the form $\mathcal{R}_{\mathbb{C}/\mathbb{R}} G'$, where G' is an absolutely simple (complex) algebraic group. In the second case, when $G = \mathcal{R}_{\mathbb{C}/\mathbb{R}}G'$, we have that $G(\mathbb{R})$ and $G'(\mathbb{C})$ are isomorphic as Lie groups. An $\mathbb{R}$-simple group is said to be of *type* A_l if it is absolutely simple of type A_l or of the form $\mathcal{R}_{\mathbb{C}/\mathbb{R}}G'$ for some absolutely simple group G' of type A_l. Similarly, we define $\mathbb{R}$-simple groups of *type* $B_l, C_l, D_l, E_6, E_7, E_8, F_4$ and G_2. A semisimple group over $\mathbb{R}$ is said to be of *classical type* if all of its $\mathbb{R}$-simple factors are one of types A_l, B_l, C_l, D_l.

An algebraic group G over $\mathbb{R}$, not necessarily reductive, is said to be $\mathbb{R}$-*quasisplit* if it admits a Borel subgroup defined over $\mathbb{R}$. In an $\mathbb{R}$-algebraic group H a unipotent element $u \in H(\mathbb{R})$ is said to be *semiregular* (see Definition 4.21, p. 242 [25]) if any semisimple element in $Z_H(u)(\mathbb{R})$ is central in H.

3. Preliminary characterization of surjectivity of P_n

In this section we obtain some preliminary results which will be used in the subsequent sections. In Theorem 3.3 we will obtain basic necessary and sufficient conditions on the surjectivity of P_n.

We first recall a result which is well-known.

Lemma 3.1. *Let $GL_n(\mathbb{C})$ be equipped with the usual $\mathbb{R}$-structure and let G be an $\mathbb{R}$-algebraic subgroup of $GL_n(\mathbb{C})$. Assume that $H \in M_n(\mathbb{R})$ be a hyperbolic semisimple element such that $\exp(H) \in G$. Then $\exp(tH) \in G(\mathbb{R})$ for all $t \in \mathbb{R}$.*

Proof. For a proof of this well-known fact see Lemma 3.6, p. 131 [4]. □

We next prove a lemma which will be needed in the proof of Theorem 3.3. It is well known that exponential map is a bijection from real symmetric matrices to positive definite real symmetric matrices. Also recall that, for an algebraic group G over $\mathbb{R}$, the exponential map $\exp : \mathcal{N}_{L(G(\mathbb{R}))} \to \mathcal{U}_{G(\mathbb{R})}$ is a bijection. In the next Lemma 3.2 we generalize these facts.

Lemma 3.2. *Let G be a complex algebraic group over $\mathbb{R}$. Then the exponential map $\exp : \mathcal{P}_{L(G(\mathbb{R}))} \to \mathcal{P}_{G(\mathbb{R})}$ is a bijection. In particular, if $g \in G(\mathbb{R})$ and $X \in \mathcal{P}_{L(G(\mathbb{R}))}$ then $g\exp(X)g^{-1} = \exp(X)$ if and only if $\mathrm{Ad}(g)X = X$.*

Proof. We first prove that $\exp : \mathcal{P}_{L(G(\mathbb{R}))} \to \mathcal{P}_{G(\mathbb{R})}$ is surjection. We choose an $\mathbb{R}$-embedding of G in $GL_n(\mathbb{C})$ which is equipped with its usual $\mathbb{R}$-structure. Let D be the diagonal subgroup of $GL_n(\mathbb{C})$. Let $g \in \mathcal{P}_{G(\mathbb{R})}$. Note that as $g_s = g_h$ and as D is a maximal $\mathbb{R}$-split torus of $GL_n(\mathbb{C})$, there is an $\alpha \in GL_n(\mathbb{R})$ such that $\alpha g_h \alpha^{-1} \in D(\mathbb{R})^*$. This implies that $g_h = \exp(H)$ for some hyperbolic element in $M_n(\mathbb{R})$. Now let U be the Zariski closure of the group generated by g_u. Observe that $Z_{G(\mathbb{R})}(U)$ is an $\mathbb{R}$-algebraic subgroup of $GL_n(\mathbb{C})$ and $\exp(H) \in Z_{G(\mathbb{R})}(U)$. Since $H \in M_n(\mathbb{R})$ is hyperbolic, by Lemma 3.1, we have that

$$\exp(tH) \in Z_{G(\mathbb{R})}(U), \quad \text{for all } t \in \mathbb{R}.$$

Let $N \in \mathcal{N}_{L(G(\mathbb{R}))}$ be such that $\exp(N) = u$. As $\exp : \mathcal{N}_{L(G(\mathbb{R}))} \to \mathcal{U}_{G(\mathbb{R})}$ is a bijection, it follows that $U(\mathbb{R}) = \{\exp(sN) \mid s \in \mathbb{R}\}$ and that $[H, N] = 0$. Hence $\exp(H + N) \in \mathcal{P}_{L(G(\mathbb{R}))}$ and $\exp(H + N) = \exp(H)\exp(N) = g_h g_u$.

We next show that $\exp : \mathcal{P}_{L(G(\mathbb{R}))} \to \mathcal{P}_{G(\mathbb{R})}$ is injection. As G is embedded in $GL_n(\mathbb{C})$ as an $\mathbb{R}$-algebraic subgroup, it is enough to prove that $\exp : \mathcal{P}_{M_n(\mathbb{R})} \to \mathcal{P}_{GL_n(\mathbb{R})}$ is injective. Let $H, I \in \mathcal{P}_{M_n(\mathbb{R})}$ be such that $\exp(H) = \exp(I)$. Then, using the uniqueness of the Jordan decomposition we have $\exp(H_h) = \exp(I_h)$ and $\exp(H_n) = \exp(I_n)$. As $\exp : \mathcal{N}_{L(G(\mathbb{R}))} \to \mathcal{U}_{G(\mathbb{R})}$ is a bijection we have $H_n = I_n$. So it remains to prove that $H_h = I_h$. By Lemma 3.1 it follows that $\exp(tH_h) \in Z_{GL_n(\mathbb{R})}(I_h)$ for all $t \in \mathbb{R}$ and consequently, $[H_h, I_h] = 0$. Thus $\exp(H_h - I_h) = 1$. Since $[H_h, I_h] = 0$ there is $\alpha \in GL_n(\mathbb{R})$ such that $\alpha(H_h - I_h)\alpha^{-1} \in L(D(\mathbb{R})^*)$. As $\exp : L(D(\mathbb{R})^*) \to D(\mathbb{R})^*$ is a bijection it follows that $\alpha(H_h - I_h)\alpha^{-1} = 0$. Thus $H_h = I_h$.

Now last part follows immediately from the first part. □

In view of Lemma 3.2, it now follows that if G is an algebraic group over $\mathbb{R}$ and A is a subgroup of $G(\mathbb{R})$ containing the connected component $G(\mathbb{R})^*$ then (complete) Jordan decomposition hold for elements in A, that is, if $g \in A$ then $g_s, g_e, g_h, g_u, g_p \in A$. Our next Theorem 3.3, on the preliminary characterization on the surjectivity of P_n, generalizes Lemma 2.3 of [7].

Theorem 3.3. *Let G be a complex algebraic group defined over $\mathbb{R}$. Let A be a subgroup with $G(\mathbb{R})^* \subset A \subset G(\mathbb{R})$. Let $g \in A$ and n be an integer. Then the following are equivalent.*

(1) $g \in P_n(A)$.
(2) $g_s \in P_n(Z_A(g_u))$.
(3) $g_e \in P_n(Z_A(g_p))$.

Consequently, $P_n : A \to A$ is surjective if and only if for every unipotent element $u \in A$ and for every semisimple element $s \in Z_A(u)(\mathbb{R})$ there is a semisimple element $t \in Z_A(u)(\mathbb{R})$ such that $s = t^n$ if and only if for every positive element $p \in A$ and for every compact element $e \in Z_A(p)(\mathbb{R})$ there is a compact element $f \in Z_A(u)(\mathbb{R})$ such that $e = f^n$.

Proof. We first prove the equivalence of (1), (2) and (3). We omit the proof of the last part as it follows easily from the equivalence of the three statements.

$1 \Leftrightarrow 2$: Suppose $g \in P_n(A)$. Then $g = h^n$ for some $h \in A$ and it follows immediately that $g_s = h_s^n$ and $g_u = h_u^n$. Clearly h_s commutes with g_u and consequently $g_s \in P_n(Z_A(g_u))$.

Now suppose $g_s \in P_n(Z_A(g_u))$. This implies that $g_s = k^n$ for some $k \in Z_A(g_u)$. Let U be the Zariski closure of the group generated by g_u. Then U is a unipotent group defined over $\mathbb{R}$ with dimension at the most one and k commutes with all the elements in U. Further, note that $g_u \in U(\mathbb{R})$. As $U(\mathbb{R})$ is isomorphic to $\mathbb{R}$ with its additive group structure, there is a $w \in U(\mathbb{R})$ such that $g_u = w^n$. As k commutes with w it follows that $g = k^n w^n = (kw)^n$.

$1 \Leftrightarrow 3$: Let $g \in P_n(A)$. Then $g = h^n$ for some $h \in A$. This implies that $g_e = h_e^n$ and $g^p = h_p^n$. By Lemma 3.2 there exist $X, Y \in \mathcal{P}_{L(A)}$ with $nY = X$ such that $g^p = \exp(X)$ and $h_p = \exp(Y)$. Again using Lemma 3.2 we conclude that $\mathrm{Ad}(h_e)X = X$ and in particular, $h_e \exp(X) h_e^{-1} = \exp(X)$. As $\exp(X) = g_p$ it follows that $h_e \in Z_A(g_p)$.

Conversely, suppose $g_e = w^n$ for some $w \in Z_A(g_p)$. Using Lemma 3.2, there is $Z \in \mathcal{P}_{L(A)}$ such that $\exp(Z) = g_p$ and $\mathrm{Ad}(w)Z = Z$. Set $v = \exp(Z/n)$. Then w commutes with v and $g = g_e g_p = (wv)^n$. □

4. Surjectivity of P_n: From complex algebraic groups to its real points

In this section we prove Theorem 1.1, Corollary 4.3, Corollary 1.2, Theorem 1.3, Corollary 1.4 and Corollary 1.5. We also give an example of a linear solvable Lie group for which no P_n, $n \geqslant 2$, is surjective; see Example 4.10. In Example 4.11 we show that the same conclusion holds when the group is the simply connected cover of $SL_2(\mathbb{R})$.

Recall that an element g in a real Lie group A is said to be *P_n-regular in A* if the linear transformation $\mathrm{Ad}(g) : L(A) \to L(A)$ does not have a nontrivial n-th root of unity in $\mathbb{C}$ as an eigenvalue.

Lemma 4.1. *Let G be a Zariski connected complex algebraic group and $s \in G$ be a semisimple element. Assume that h is an n-th root of s, that is, $s = h^n$. Then h is P_n regular if and only if $h \in Z(Z_G(s)^0)$.*

Proof. Suppose h is P_n-regular. Let $X \in L(Z_G(s)^0)$. As $h^n = s$ we have $\mathrm{Ad}(h^n)X = X$. Now as h is P_n regular it follows that $\mathrm{Ad}(h)X = X$. Thus $h \in Z(Z_G(s)^0)$. Now suppose $h \in Z(Z_G(s)^0)$ and $h^n = s$. Let $\lambda \in \mathbb{C}^*$ be an eigenvalue of $\mathrm{Ad}(h)$ with $\lambda^n = 1$. Then there is $X \in L(G)$, $X \neq 0$ so that $\mathrm{Ad}(h)X = \lambda X$. Hence $\mathrm{Ad}(s)X = X$ and consequently $X \in L(Z_G(s)^0)$. But as $h \in Z(Z_G(s)^0)$ we have $\mathrm{Ad}(h)X = X$. Thus $\lambda = 1$. □

We need to recall the two following results from [7].

Theorem 4.2. *(See [7].) Let G be a Zariski connected algebraic group over an algebraically closed field of characteristic zero and let n be a natural number. Then the following are equivalent.*

(1) *$P_n : G \to G$ is surjective.*
(2) *$P_n : Z_G(s)^0 \to Z_G(s)^0$ is surjective for every semisimple element $s \in G$.*
(3) *$P_n : Z_G(u) \to Z_G(u)$ is surjective for every unipotent element $u \in G$.*

See [7, Corollary B] for a proof of the above theorem.

Theorem 4.3. *(See [7].) Let G be a Zariski connected algebraic group over an algebraically closed field of characteristic zero and let n be a natural number. Suppose $P_n : G \to G$ is surjective then $P_n : Z(G) \to Z(G)$ is surjective.*

See [7, Corollary 3.5] for a proof of the above theorem.

Theorem 4.4. *Let G be a Zariski connected complex algebraic group defined over $\mathbb{R}$. Let n be an odd integer and $P_n : G \to G$ be surjective. Then for every semisimple element $s \in G(\mathbb{R})$ there is a P_n-regular element h in G so that $h \in G(\mathbb{R})$ and $s = h^n$.*

Proof. Suppose $P_n : G \to G$ is surjective where n is an odd integer. Let $s \in G(\mathbb{R})$ be a semisimple element. In view of Lemma 4.1 it is enough to prove that $P_n : Z(Z_G(s)^0)(\mathbb{R}) \to Z(Z_G(s)^0)(\mathbb{R})$ is surjective. Let $\sigma : G \to G$ be the anti-holomorphic automorphism of G so that $G(\mathbb{R})$ is precisely the fixed points of this automorphism. By Theorem 4.2 we conclude that $P_n : Z_G(s)^0 \to Z_G(s)^0$ is surjective. Hence by Theorem 4.3 we see that $P_n : Z(Z_G(s)^0) \to Z(Z_G(s)^0)$ is surjective. Let $r \in Z(Z_G(s)^0)$ be such that $r^n = s$. Now as $s \in G(\mathbb{R})$ it follows that $\sigma(r)^n = \sigma(s) = s$. Note that as $s \in G(\mathbb{R})$ the group $Z_G(s)^0$ is defined over $\mathbb{R}$. Hence $Z(Z_G(s)^0)$ is also defined over $\mathbb{R}$ and consequently $\sigma(r) \in Z(Z_G(s)^0)$. Now as both r and $\sigma(r)$ lie in the center of the group $Z_G(s)^0$, we conclude that r, $\sigma(r)$ and s commute. Hence $(r\sigma(r))^n = s^2$. As n is an odd integer $n + 2m = 1$ for some integer m. Now

$$s = s^{n+2m} = s^n s^{2m} = s^n \big(r\sigma(r)\big)^{mn} = \big(sr^m\sigma(r)^m\big)^n.$$

Now clearly $\sigma(sr^m\sigma(r)^m) = sr^m\sigma(r)^m$ and hence $sr^m\sigma(r)^m \in Z(Z_G(s)^0)(\mathbb{R})$. □

Lemma 4.5. *Let T be an $\mathbb{R}$-anisotropic torus. Then $T(\mathbb{R})$ is the maximal compact subgroup of $T = T(\mathbb{C})$, when considered as a real Lie group. Further, as a (real) Lie group, T is the direct product of $T(\mathbb{R})$ and a vector subgroup V.*

Proof. Recall that as a complex algebraic group T is the direct product of n copies of $\mathbb{C}^*$ where n is the dimension of T as an algebraic group. Hence the unique maximal compact subgroup of T is connected and its dimension as a real Lie group is n. Now the proof is completed by observing that $T(\mathbb{R})$ compact, connected and has the dimension n as a real Lie group. The second assertion is obvious now. $\square$

Lemma 4.6. *Let G be a complex algebraic group over $\mathbb{R}$ and let T be a maximal torus defined over $\mathbb{R}$ which is $\mathbb{R}$-anisotropic. Let $X \in L(G)$. Then $P_n : Z_T(X) \to Z_T(X)$ is surjective if and only if $P_n : Z_{T(\mathbb{R})}(X) \to Z_{T(\mathbb{R})}(X)$ is surjective.*

Proof. Let $\lambda_1, \ldots, \lambda_l$ be the l distinct complex characters of T such that $L(G)$ can be decomposed as a direct sum of the weight spaces, $L(G) = \sum_i L(G)_{\lambda_i}$ where $L(G)_{\lambda_i}$ is the weight space, corresponding to the character λ_i, under the adjoint action of T on $L(G)$.

By Lemma 4.5, $T(\mathbb{R})$ is the maximal compact subgroup of T. Consequently, there exists a subgroup V, isomorphic to $\mathbb{R}^n$, of T such that for any complex character μ of T, $\mu(v)$ is a positive real number for all $v \in V$ and $|\mu(t)| = 1$ for all $t \in T(\mathbb{R})$. Let $X \in L(G)$. We claim that $Z_T(X)$ is a direct product of the subgroups $Z_{T(\mathbb{R})}(X)$ and $Z_V(X)$. It is enough to show that if $\alpha \in T(\mathbb{R})$, $\beta \in V$ be such that $\alpha\beta \in Z_T(X)$ then $\alpha \in Z_{T(\mathbb{R})}X$ and $\beta \in Z_V(X)$. Note that $X = \sum_i X_i$ for some $X_i \in L(G)_{\lambda_i}$. As $\mathrm{Ad}(\alpha\beta)X = X$ we have $\mathrm{Ad}(\alpha\beta)X_i = X_i$, for all i which further implies that $\lambda_i(\alpha\beta)X_i = X_i$. Thus for those i, for which $X_i \neq 0$, one has $\lambda_i(\alpha\beta) = 1$. Now as $|\lambda_i(\alpha)| = 1$ and $\lambda_i(\beta)$ is a positive real we conclude that for those i, for which $X_i \neq 0$, $\lambda_i(\alpha) = \lambda_i(\beta) = 1$. Thus, for those i, for which $X_i \neq 0$, we have $\mathrm{Ad}(\alpha)X_i = \mathrm{Ad}(\beta)X_i = X_i$. Hence $\mathrm{Ad}(\alpha)X = \mathrm{Ad}(\beta)X = X$. In other words, $\alpha \in Z_{T(\mathbb{R})}X$ and $\beta \in Z_V(X)$, which is what we wanted to prove.

We next see that $Z_V(X)$ is a connected group. As $Z_T(X)$ is a direct product of the subgroups $Z_{T(\mathbb{R})}(X)$ and $Z_V(X)$ and as the number of connected components of $Z_T(X)$ is finite, we have that number of connected components of $Z_V(X)$ is finite. But $Z_V(X)$ is closed subgroup of the vector group V and hence $Z_V(X)$ is connected.

As $Z_T(X)$ is a direct product of the subgroups $Z_{T(\mathbb{R})}(X)$ and $Z_V(X)$ where $Z_V(X)$ is connected, abelian it is now clear that $P_n : Z_T(X) \to Z_T(X)$ is surjective if and only if $P_n : Z_{T(\mathbb{R})}(X) \to Z_{T(\mathbb{R})}(X)$. $\square$

Lemma 4.7. *Let G be a Zariski connected complex algebraic group and let T be a maximal torus. Then $P_n : Z_T(X) \to Z_T(X)$ is surjective for all nilpotent $X \in L(G)$ if and only if $P_n : Z_T(Y) \to Z_T(Y)$ is surjective for all $Y \in L(G)$.*

Proof. We will show that if $P_n : Z_T(X) \to Z_T(X)$ is surjective for all nilpotent elements $X \in L(G)$, then $P_n : Z_T(Y) \to Z_T(Y)$ is surjective for all $Y \in L(G)$. In fact, this will follow at once from the observation that for any $Y \in L(G)$ there is a nilpotent $X \in L(G)$ such that $Z_T(Y) = Z_T(X)$. We will prove this observation. Let F be a reductive Levi factor of G, containing T. Let

B be a Borel subgroup of F containing T. Let U be the unipotent radical of F and let U^- be the unipotent radical of the Borel subgroup opposite to B. Then

$$L(G) = L\big(U^-\big) + L(T) + L(U) + L(R_u G).$$

For $Y \in L(G)$, there exist $Y^- \in L(U^-)$, $Y^0 \in L(T)$, $Y^+ \in L(U)$ and $Z \in L(R_u G)$ such that $Y = Y^- + Y^0 + Y^+ + Z$. Let $X(T)$ be the group of characters of T and let $\Delta \subset X(T)$ be the finite set of roots so that $L(U) = \sum_{\chi\in\Delta} L(F)_\chi$, where $L(F)_\chi$ is the root space, corresponding to the character χ, in the adjoint representation of T on $L(F)$. Then clearly, $L(U^-) = \sum_{\chi\in-\Delta} L(F)_\chi$. We now choose $W \in L(U)$ in such a way that, for all $\chi \in \Delta$, the $L(F)_\chi$-component in W is non-zero if and only if either the $L(F)_\chi$-component of Y^+ is non-zero or $L(F)_{-\chi}$-component of Y^- is non-zero. This implies that, $Z_T(Y^+ + Y^-) = Z_T(W)$. Further, as T centralizes all of $L(T)$, it now follows that $Z_T(Y) = Z_T(W + Z)$. But $W + Z \in L(U) + L(R_u G)$ and $L(U) + L(R_u G)$ is the Lie algebra of the unipotent group $U R_u G$. Hence $W + Z$ is a nilpotent element in $L(G)$. This completes the proof. $\square$

Lemma 4.8. *Let G be a Zariski connected complex algebraic group and H be a Zariski connected algebraic subgroup of maximal rank. If $P_n : G \to G$ is surjective then $P_n : H \to H$ is surjective.*

Proof. Note that a maximal torus of H remains to be a maximal torus of G. Now we apply Theorem A [7], characterizing the surjectivity of P_n, to complete the proof. $\square$

Proof of Theorem 1.1. Suppose $P_n : G \to G$ is surjective. Then, by Theorem 1.4 of [10], this is equivalent to saying that $P_n : G^0 \to G^0$ is surjective and n is coprime to the number of (Zariski) connected components of G. Let H be a maximal rank algebraic subgroup of G. Then by Lemma 4.8 it is immediate that $P_n : H^0 \to H^0$ is surjective. With this observation we can now proceed for a proof, given in steps.

(1): As H is Zariski connected it follows that $P_n : H \to H$ is surjective. Let T be a maximal torus of H defined over $\mathbb{R}$ and $X \in \mathcal{N}_{L(G(\mathbb{R}))}$. Let $t \in Z_{T(\mathbb{R})}(X)$. Assuming n to be odd we apply Theorem 4.4 and get $s \in H(\mathbb{R})$ such that s is P_n-regular and $t = s^n$. Now as $\mathrm{Ad}(t)X = X$ and $\mathrm{Ad}(t)W = W$ for all $W \in L(T)$ and as s is P_n-regular with $t = s^n$ it follows that $\mathrm{Ad}(s)X = X$ and $\mathrm{Ad}(s)W = W$ for all $W \in L(T)$. This implies that $s \in Z_G(T) \cap Z_G(X)$. Note that semisimple elements of $Z_G(T)$ lie in T. Thus $s \in Z_{T(\mathbb{R})}(X)$.

(2): Recall that, as $P_n : G \to G$ is surjective, $P_n : H^0 \to H^0$ is also surjective. We first show that $P_n : H^0(\mathbb{R}) \to H^0(\mathbb{R})$ is surjective. Let $u \in H^0(\mathbb{R})$ be a unipotent element and $s \in Z_G(u)(\mathbb{R})$ be a semisimple element. As n is odd, by Theorem 4.4 there is a P_n-regular element $h \in H^0$ so that $h \in H^0(\mathbb{R})$ and $s = h^n$. Note that $u \in Z_{H^0}(s)^0$ and by Lemma 4.1 $h \in Z(Z_{H^0}(s)^0)$. Thus $h \in Z_{H^0}(u)(\mathbb{R})$. Hence appealing to the equivalence of (1) and (2) in Theorem 3.3 we see that $g \in P_n(H^0(\mathbb{R}))$. Thus $P_n : H^0(\mathbb{R}) \to H^0(\mathbb{R})$ is surjective. As $H(\mathbb{R})^* = H^0(\mathbb{R})^*$ it follows, by Theorem 1.8 of [10], that $P_n : H(\mathbb{R})^* \to H(\mathbb{R})^*$ is surjective.

(3): Let G be an algebraic group over $\mathbb{R}$ and H be an $\mathbb{R}$-algebraic subgroup of maximal rank. Further assume that H contains a maximal torus T, which is defined over $\mathbb{R}$ and $\mathbb{R}$-anisotropic.

Note that $T(\mathbb{R})$ is connected and hence lies in $H(\mathbb{R})^*$. Let K be a maximal compact subgroup of $H(\mathbb{R})^*$ containing $T(\mathbb{R})$. It is then clear that $T(\mathbb{R})$ is a maximal (compact) torus of K. Recall that any compact element lies in a conjugate of K and further, it lies in a conjugate of $T(\mathbb{R})$.

We recall that, as $P_n : G \to G$ is surjective, by Theorem 1.4 of [10] and Lemma 4.8, $P_n : H^0 \to H^0$ is also surjective. Let $g \in H(\mathbb{R})^*$. Let $g = g_e g_h g_u$ be the complete Jordan decomposition of $g \in G(\mathbb{R})$. It follows that $g_e, g_h, g_u \in H(\mathbb{R})^*$. Note that there is an $\alpha \in H(\mathbb{R})^*$ such that $\alpha g_e \alpha^{-1} \in T(\mathbb{R})$. We will show that $\alpha g \alpha^{-1}$ has an n-th root in $H(\mathbb{R})^*$. Let $\alpha g \alpha^{-1} = t$. Then $t_e = \alpha g_e \alpha^{-1}$ and $t_p = \alpha g_p \alpha^{-1}$. By the equivalence of (1) and (3) in Theorem 3.3 it is enough to show that $t_e \in P_n(Z_{H(\mathbb{R})^*}(t_p))$. Using Lemma 3.2 we get that $t_p = \exp(Y)$ for some $Y \in \mathcal{P}_{L(H(\mathbb{R})^*)}$ and that $Z_{H(\mathbb{R})^*}(t_p) = Z_{H(\mathbb{R})^*}(Y)$. As $P_n : H^0 \to H^0$ is surjective, it follows from Theorem A in [7] that $P_n : Z_T(W') \to Z_T(W')$ is surjective for all nilpotent elements $W' \in L(H^0)$. By Lemma 4.7 this implies that $P_n : Z_T(W) \to Z_T(W)$ is surjective for all $W \in L(H^0)$. As T is $\mathbb{R}$-anisotropic, by Lemma 4.6, it follows that $P_n : Z_{T(\mathbb{R})}(W) \to Z_{T(\mathbb{R})}(W)$ is surjective for all $W \in L(H^0)$. Thus $P_n : Z_{T(\mathbb{R})}(W) \to Z_{T(\mathbb{R})}(W)$ is surjective for all $W \in L(H(\mathbb{R})^*)$. In particular, as $t_e \in Z_{T(\mathbb{R})}(Y)$ there is a compact element $r \in Z_{T(\mathbb{R})}(Y)$ such that $t = r^n$. Clearly $r \in Z_{H(\mathbb{R})^*}(Y)$. This completes the proof.

We now prove the last part of the theorem using Theorem 4.4. By hypothesis n is odd and by Lemma 4.8, as H is Zariski connected, $P_n : H \to H$ is surjective. As $Z(H(\mathbb{R}))$ is abelian, it is enough to prove that for every semisimple element $s \in Z(H(\mathbb{R}))$ we have $s \in P_n(Z(H(\mathbb{R})))$. As n is odd, by Theorem 4.4, there is a $t \in H(\mathbb{R})$ such that t is P_n-regular in H and $s = t^n$. As $\mathrm{Ad}(s)X = X$ for all $X \in L(H)$ we then have $\mathrm{Ad}(t)X = X$ for all $X \in L(H)$. Thus, as H is Zariski connected, $t \in Z(H)$ which in turn implies $t \in Z(H(\mathbb{R}))$. □

Remark 4.9. Note that Theorem 1.1 implies that if G is an algebraic group, n is odd and $P_n : G \to G$ is surjective then $P_n : G(\mathbb{R}) \to G(\mathbb{R})$ is surjective. We remark that the converse of this result is not true as can be seen easily. Let m be an odd integer. Then $P_n : SL_m(\mathbb{C}) \to SL_m(\mathbb{C})$ is surjective if and only if m is relatively prime to n (see Theorem 4.15), but it can be seen (using (1) of Theorem 1.3) that $P_n : SL_m(\mathbb{R}) \to SL_m(\mathbb{R})$ is surjective if and only if n is odd.

Proof of Corollary 1.2. Let G be a complex algebraic group defined over $\mathbb{R}$. Then using Corollary 5.1 of [7] and Theorem 1.4 of [10] we conclude that there is an integer k'_G such that if n is coprime to k'_G then $P_n : G \to G$ is surjective. Set $k_G = 2k'_G$. Let now n be coprime to k_G then n is odd and $P_n : G \to G$ is surjective. Then by (2) of Theorem 1.1, $P_n : G(\mathbb{R})^* \to G(\mathbb{R})^*$ is surjective. We now recall that the order $\mathrm{Ord}(G(\mathbb{R})/G(\mathbb{R})^*)$ is finite. Let $m_G = k_G \cdot \mathrm{Ord}(G(\mathbb{R})/G(\mathbb{R})^*)$. If n is coprime to m_G then n is coprime to $\mathrm{Ord}(G(\mathbb{R})/G(\mathbb{R})^*)$ and $P_n : G(\mathbb{R})^* \to G(\mathbb{R})^*$ is surjective. Theorem 1.8 of [10] says that for an algebraic group H defined over $\mathbb{R}$, the map $P_n : H(\mathbb{R}) \to H(\mathbb{R})$ is surjective if and only if $P_n : H(\mathbb{R})^* \to H(\mathbb{R})^*$ is surjective and n is coprime to the order $\mathrm{Ord}(H(\mathbb{R})/H(\mathbb{R})^*)$. We now use this result to conclude that if n is coprime to m_G then $P_n : G(\mathbb{R}) \to G(\mathbb{R})$ is surjective. □

We now give two examples to show that Corollary 1.2 does not hold in general.

Example 4.10. Let A be a diagonalizable matrix in $M_n(\mathbb{C})$ with $\det A \neq 0$. Let us consider the linear Lie group G defined by

$$G = \left\{ \begin{pmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \exp(tA) & v \\ 0 & 0 & 0 & 1 \end{pmatrix} \,\middle|\, t \in \mathbb{R},\ v \in \mathbb{C}^n \right\}.$$

Then G is a closed subgroup of $GL_{n+3}(\mathbb{C})$ which is also a solvable and simply connected Lie group isomorphic to the semidirect product $\mathbb{R} \ltimes_\rho \mathbb{C}^n$, where the homomorphism $\rho : \mathbb{R} \to GL_n(\mathbb{C})$ is given by $\rho(t) = \exp(tA)$, $t \in \mathbb{R}$.

Claim. *$P_n : G \to G$ is not surjective for all n if and only if A has an eigenvalue which is purely imaginary, that is, in $\sqrt{-1}\,\mathbb{R} \smallsetminus \{0\}$.*

Let

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & tA & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}.$$

Observe that, as A is a diagonalizable matrix, $\mathfrak{h}$ is contained in a Cartan subalgebra of $L(G)$. Further, as $\det A \neq 0$, we conclude that $\mathfrak{h}$ itself is a Cartan subalgebra of $L(G)$. As G is connected solvable, Cartan subgroups of G are connected and consequently the subgroup H defined by

$$H = \exp(\mathfrak{h}) = \left\{ \begin{pmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \exp(tA) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}$$

is a Cartan subgroup of G. Let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues of A and $\mathbb{C}^n = \mathbb{C}^n_{\lambda_1} + \cdots + \mathbb{C}^n_{\lambda_k}$ be the eigenspace decomposition of $\mathbb{C}^n$ where $\mathbb{C}^n_{\lambda_i} = \{v \in \mathbb{C}^n \mid Av = \lambda_i v\}$. We next describe $Z_H(v)$ for all $v \in \mathbb{C}^n$. Let $p_i : \mathbb{C}^n \to \mathbb{C}^n_{\lambda_i}$ be the i-th projection map with respect to the above decomposition of $\mathbb{C}^n$. For better clarity of the exposition we introduce the notion of *support of* v by $\operatorname{Supp}(v) = \{i \mid p_i(v) \neq 0\}$. Then it is easy to see that, for $v \in \mathbb{C}^n$, the group $Z_H(v)$ is isomorphic to

$$\bigcap_{i \in \operatorname{Supp}(v)} \left(\frac{2\pi\sqrt{-1}}{\lambda_i} \mathbb{Z} \right) \cap \mathbb{R}.$$

In view of this, there exists $v \in \mathbb{C}^n$ such that $Z_H(v)$ is isomorphic to the additive group of integers $\mathbb{Z}$ if and only if A has an eigenvalue in $\sqrt{-1}\,\mathbb{R} \smallsetminus \{0\}$. We now appeal to Theorem A of [6] to complete the proof of the above claim. Alternatively, as G is simply connected and solvable, we may also use Theorem C of [6] to reach the same conclusion.

Example 4.11. Let $\widetilde{SL_2(\mathbb{R})}$ be the simply connected cover of $SL_2(\mathbb{R})$. We claim that for all $n \geqslant 2$ the map $P_n : \widetilde{SL_2(\mathbb{R})} \to \widetilde{SL_2(\mathbb{R})}$ is not surjective. We will first sketch that if G is an algebraic group over $\mathbb{R}$ and if H is a covering group of the real Lie group $G(\mathbb{R})^*$ with covering map $\eta : H \to G(\mathbb{R})^*$ then an analogue of the Jordan decomposition is available in the Lie group H. This is also done in Proposition 2.2 of [3] with the additional assumption that $\eta : H \to G(\mathbb{R})^*$ is a finite covering map. Recalling that $\mathrm{d}\eta : L(H) \to L(G(\mathbb{R})^*)$ is an isomorphism we define the set of unipotent elements $\mathcal{U}_H$ of H by $\mathcal{U}_H = \exp_H \circ\, \mathrm{d}\eta^{-1}(\mathcal{N}_{L(G(\mathbb{R})^*)})$. It is not difficult to see that $\eta(\mathcal{U}_H) = \mathcal{U}_{G(\mathbb{R})^*}$ and that $\eta : \mathcal{U}_H \to \mathcal{U}_{G(\mathbb{R})^*}$ is a bijection. Using these facts we observe that for each $g \in H$ there is a unique pair $g_s, g_u \in H$ such that $g = g_s g_u = g_u g_s$ with $g_u \in \mathcal{U}_H$

and that $\eta(g_s)$ (resp. $\eta(g_u)$) is the semisimple part of $\eta(g)$ (resp. the unipotent part of $\eta(g_u)$). We call g_s and g_u the semisimple and unipotent parts of g respectively. It is easy to see that if $\mathcal{U}_{G(\mathbb{R})^*} \cap Z(G(\mathbb{R})^*) = e$ then $Z(H)$ consists of semisimple elements. We are now in a position to prove the following claim.

Claim. *Let now $G = SL_2(\mathbb{C})$ be equipped with the usual $\mathbb{R}$-structure so that $G(\mathbb{R}) = SL_2(\mathbb{R})$ and let H be a covering group of $G(\mathbb{R})^* = G(\mathbb{R})$. Assume that $P_n : H \to H$ is surjective then $P_n : Z(H) \to Z(H)$ is surjective.*

As before let $\eta : H \to SL_2(\mathbb{R})$ be the covering map. Also by the facts stated above, $Z(H)$ consists of semisimple elements. Let u be the unipotent element in $\mathcal{U}_{SL_2(\mathbb{R})}$ defined by $u = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Let $\tilde{u}$ be the unique element in $\mathcal{U}_H$ such that $\eta(\tilde{u}) = u$. Let $\alpha \in Z(H)$. If $g = \alpha\tilde{u}$ then it is immediate that $g_s = \alpha$ and $g_u = \tilde{u}$. It is a fact that u is a regular unipotent element in $SL_2(\mathbb{C})$ and consequently all the semisimple elements of $Z_{SL_2(\mathbb{R})}(u)$ are central in $SL_2(\mathbb{R})$ (compare this with Theorem 1.8). As $P_n : H \to H$ is surjective there is $h \in H$ such that $g = h^n$. Clearly $\alpha = h_s^n$ and $\tilde{u} = h_u^n$. We now show that $h_s \in Z(H)$. Note that $\eta(h_s)$ is semisimple and commutes with u. This forces $\eta(h_s) \in Z(SL_2(\mathbb{R}))$. Thus $h_s \in Z(H)$. This completes the proof of the claim.

Now if $H = \widetilde{SL_2(\mathbb{R})}$ then, by the above claim, surjectivity of $P_n : \widetilde{SL_2(\mathbb{R})} \to \widetilde{SL_2(\mathbb{R})}$ implies surjectivity of $P_n : Z(\widetilde{SL_2(\mathbb{R})}) \to Z(\widetilde{SL_2(\mathbb{R})})$. But $Z(\widetilde{SL_2(\mathbb{R})})$ is isomorphic to the additive group of integers $\mathbb{Z}$. Thus $P_n : \widetilde{SL_2(\mathbb{R})} \to \widetilde{SL_2(\mathbb{R})}$ is not surjective if $n \neq 1$.

Remark 4.12. In a similar vein, as in Example 4.11, one may consider the questions of surjectivity of the exponential map $\exp : L(\widetilde{SL_2(\mathbb{R})}) \to \widetilde{SL_2(\mathbb{R})}$. This question and related structures are studied in depth in [18] where the authors show that the image $\exp(L(\widetilde{SL_2(\mathbb{R})}))$ misses a nonempty open set in $\widetilde{SL_2(\mathbb{R})}$ and thus, $\exp(L(\widetilde{SL_2(\mathbb{R})}))$ is not even dense in $\widetilde{SL_2(\mathbb{R})}$. In fact, in p. 423 [18] the authors employ a very interesting parametrization of $\widetilde{SL_2(\mathbb{R})}$ by $\mathbb{R}^3$ and in p. 429 [18] and in p. 18 [20] elegant pictorial descriptions of the set $\exp(L(\widetilde{SL_2(\mathbb{R})}))$ in $\mathbb{R}^3$ are given enabling one to visualize the above fact.

We now proceed towards a proof of Theorem 1.3. We first need a lemma.

Lemma 4.13. *Let H be a Zariski connected complex algebraic group defined over $\mathbb{R}$ and let H_1 be a normal $\mathbb{R}$-algebraic subgroup of H which is not necessarily Zariski connected. Assume that n is odd and that $P_n : H/H_1 \to H/H_1$ is surjective. Then $P_n : H(\mathbb{R})/H_1(\mathbb{R}) \to H(\mathbb{R})/H_1(\mathbb{R})$ is surjective.*

Proof. Let $F = H/H_1$. Then by hypothesis $P_n : F \to F$ is surjective. Now as n is odd and F is Zariski connected it follows from Theorem 1.1 that $P_n : F(\mathbb{R}) \to F(\mathbb{R})$ is surjective. We next note that $H(\mathbb{R})/H_1(\mathbb{R})$ embeds in $F(\mathbb{R})$. It can be easily seen that $(H(\mathbb{R})/H_1(\mathbb{R}))^* = F(\mathbb{R})^*$. Thus we have the following inclusion

$$\big(H(\mathbb{R})/H_1(\mathbb{R})\big)^* = F(\mathbb{R})^* \subset H(\mathbb{R})/H_1(\mathbb{R}) \subset F(\mathbb{R}).$$

As F is Zariski connected, the index of $F(\mathbb{R})^*$ in $F(\mathbb{R})$ is of the form 2^l for some integer l. We recall that, by Theorem 1.7 of [10], if G is an algebraic group defined over $\mathbb{R}$, A is a subgroup of $G(\mathbb{R})$ with $G(\mathbb{R})^* \subset A \subset G(\mathbb{R})$ then $P_n : A \to A$ is surjective if and only if so is P_n :

$G(\mathbb{R})^* \to G(\mathbb{R})^*$ and n is relatively prime to the order $\mathrm{Ord}(A/G(\mathbb{R})^*)$. This result implies that $P_n : F(\mathbb{R})^* \to F(\mathbb{R})^*$ is surjective. Now, as the index of $F(\mathbb{R})^*$ in $H(\mathbb{R})/H_1(\mathbb{R})$ is of the form 2^i, applying Theorem 1.7 of [10] again, we conclude that $P_n : H(\mathbb{R})/H_1(\mathbb{R}) \to H(\mathbb{R})/H_1(\mathbb{R})$ is surjective. $\square$

Lemma 4.14. *Let G be a Zariski connected complex reductive algebraic group and H be a Zariski connected algebraic subgroup of maximal rank. Assume that $P_n : G/Z(G) \to G/Z(G)$ is surjective. Then $P_n : H/Z(H) \to H/Z(H)$ is surjective.*

Proof. As H is of maximal rank and G is reductive, $Z(G) \subset H$ and consequently $Z(G) \subset Z(H) \subset H$. Observe that $H/Z(G)$ is a maximal rank subgroup of $G/Z(G)$. Thus by Theorem A of [7], as $P_n : G/Z(G) \to G/Z(G)$ is surjective, it follows that $P_n : H/Z(G) \to H/Z(G)$. Clearly $H/Z(H)$ is a quotient of $H/Z(G)$. Hence $P_n : H/Z(H) \to H/Z(H)$ is surjective. $\square$

We now need to recall certain results of [7]. Let G be a connected simple algebraic group. We consider the root system associated to G with respect to some maximal torus of G. Let Δ be the set of simple roots with respect to an order in the root system and let $h = \sum_{\alpha \in \Delta} m_\alpha \alpha$ be the *highest root*. A prime p is said to be a *bad prime* for the simple group G if p divides m_α for some $\alpha \in \Delta$. Now if G is connected semisimple then a prime p is said to be a *bad prime* for G if p is bad for some simple factor of G.

Theorem 4.15. *(See [7].) Let G be a Zariski connected semisimple algebraic group over an algebraically closed field of characteristic zero. Then $P_n : G \to G$ is surjective if and only if n is prime to the bad primes for G and* $\mathrm{Ord}(Z(G))$. *In particular, if G is a connected simple algebraic group then $P_n : G \to G$ is surjective if and only if one of the following conditions holds (depending on the type of G).*

(1) *G is of type A_l, $l \geqslant 1$ and n is prime to* $\mathrm{Ord}(Z(G))$.
(2) *G is of type either B_l, $l \geqslant 2$ or C_l, $l \geqslant 3$ or D_l, $l \geqslant 4$, and n is prime to* 2.
(3) *G is of type either E_6 or E_7 or F_4 or G_2, and n is prime to* 6.
(4) *G is of type E_8 and n is prime to* 30.

Hence for any semisimple algebraic group G the map P_n is surjective if n is prime to $30 \cdot \mathrm{Ord}(Z(G))$.

See [7, Theorem C] for a proof of the above theorem.

Proof of Theorem 1.3. (1): Let G be a semisimple algebraic group over $\mathbb{R}$ which is of classical type. Then by $G/Z(G)$ is a direct product of adjoint groups of type either A_l or B_l or C_l or D_l. Now by (1) of Theorem 4.15, for adjoint groups of type A_l, the map P_n is surjective for all n. Further, by (2) of Theorem 4.15, for groups of type B_l or C_l or D_l the map P_n is surjective for all odd n. Hence $P_n : G/Z(G) \to G/Z(G)$ is surjective for all odd n. Let now H be an $\mathbb{R}$-algebraic Zariski connected subgroup of G of maximal rank. By Lemma 4.14 $P_n : H/Z(H) \to H/Z(H)$ is surjective for all odd n. We now appeal to Lemma 4.13 to see that $P_n : H(\mathbb{R})/Z(H(\mathbb{R})) \to H(\mathbb{R})/Z(H(\mathbb{R}))$ is surjective for all odd n. It is now immediate that $P_n : H(\mathbb{R}) \to H(\mathbb{R})$ is surjective if n is odd and $P_n : Z(H(\mathbb{R})) \to Z(H(\mathbb{R}))$ is surjective. But as

$Z(H(\mathbb{R}))$ is abelian $P_n : Z(H(\mathbb{R})) \to Z(H(\mathbb{R}))$ is surjective if and only if n is coprime to the order $\mathrm{Ord}(Z(H(\mathbb{R}))/Z(H(\mathbb{R}))^*)$. This completes the proof of the first part of (1).

To give a proof of the second part of (1), first observe that, if G is of type B_l or C_l or D_l the map P_n is surjective for all odd n (see (2) of Theorem 4.15). Consequently, as H is maximal rank, by the last part of Theorem 1.1 it is immediate that $P_n : Z(H(\mathbb{R})) \to Z(H(\mathbb{R}))$ is surjective for all odd n. We now apply the first part of (1), Theorem 1.3 to conclude the proof.

(2) and (3): The proofs of the statements follow immediately from Theorem 1.1 and (3) and (4) of Theorem 4.15.

We now give the proofs of the final parts of Theorem 1.3.

If G is an algebraic group over $\mathbb{R}$, which is semisimple and of classical type then it follows immediately from (1) that $P_n : G(\mathbb{R}) \to G(\mathbb{R})$ is surjective if n is coprime to $2 \cdot \mathrm{Ord}(Z(G(\mathbb{R})))$.

The proof of the subsequent statement follows similarly as in the proof of (1). We apply Lemmas 4.13, 4.14, after we observe, using Theorem 4.15, that if G is any semisimple group over $\mathbb{R}$ then $P_n : G/Z(G) \to G/Z(G)$ is surjective if n is coprime to 30. Thus the proof of the theorem is completed. □

We now need to recall a criterion, which follows from results of K. Hofmann and J. Lawson in [19] and M. McCrudden in [21], on the exponentiality of real Lie group.

Theorem 4.16. *Let G be a closed subgroup of a real Lie group which is not necessarily connected. Then $G = \exp(L(G))$ if and only if $P_n : G \to G$ is surjective for all $n \geqslant 2$.*

Proof. The theorem follows immediately from the result that $\exp(L(G)) = \bigcap_{n\geqslant 2} P_n(G)$, which is readily implied either by Theorem A of [19] or by combination of Proposition 1 of [19] and Theorem 2 of [21]. □

Proof of Corollary 1.4. (1): By (2) of Theorem 1.1, as H is Zariski connected maximal rank subgroup of G and as $P_n : G \to G$ is surjective for all odd n, it follows that $P_n : H(\mathbb{R}) \to H(\mathbb{R})$ is surjective for all odd n. Thus $P_n : H(\mathbb{R}) \to H(\mathbb{R})$ is surjective for all n if and only if $P_2 : H(\mathbb{R}) \to H(\mathbb{R})$ is surjective. Appealing to Theorem 4.16 we conclude that $H(\mathbb{R})$ is exponential if and only if $P_2 : H(\mathbb{R}) \to H(\mathbb{R})$ is surjective.

(2): As G is exponential, $P_n : G \to G$ is surjective for all n. Consequently, by (3) of Theorem 1.1, as H is of maximal rank and contains an $\mathbb{R}$-anisotropic maximal torus, we have that $P_n : H(\mathbb{R})^* \to H(\mathbb{R})^*$ is surjective for all n. Now we apply Theorem 4.16 to conclude that $H(\mathbb{R})^*$ is exponential.

We now give a proof of the last part. As $H(\mathbb{R})$ is evenly centered, by (1) of Theorem 1.3, it follows that $P_n : H(\mathbb{R}) \to H(\mathbb{R})$ is surjective for all odd n. Now the proof follows, as above, using Theorem 4.16. □

Proof of Corollary 1.5. (1): Proofs of the statements follow immediately from (1) of Theorem 1.3, the last part of Corollary 1.4 and the fact that for any P (as in the statement of Corollary 1.5) $\mathrm{Ord}(Z(P(\mathbb{R}))) = \mathrm{Ord}(Z(SL_m(\mathbb{H}))) = 2$.

(2): First recall that $G = GL_m(\mathbb{C})$ (resp. $G = PSL_m(\mathbb{C})$) is an exponential group. So $P_n : G \to G$ is surjective for all n. As above, we now use (1) of Corollary 1.4 to see that $H(\mathbb{R})^*$ is exponential if and only if $P_2 : H(\mathbb{R}) \to H(\mathbb{R})$ is surjective.

For the proof of the last part of (2) we observe that Zariski closure of D (or any of its $G(\mathbb{R})$-conjugates) is a maximal torus of G which is defined over $\mathbb{R}$ and $\mathbb{R}$-anisotropic. The rest of the proof now follows from (2) of Corollary 1.4. □

5. Types, conjugates of elements and parabolic subgroups

In this section we prove Theorem 1.6. In order to prove Theorem 1.6 (and Theorems 1.7, 1.9 in Section 6) we need to prove Theorem 5.11 which gives certain criteria to decide when a $G(\mathbb{R})$-conjugate of $g \in G(\mathbb{R})$ will lie in some canonical solvable subgroups defined over $\mathbb{R}$ which in turn lie in standard parabolic subgroups.

We begin with the following lemma.

Lemma 5.1. *Let G be a Zariski connected reductive group over an algebraically closed field of characteristic zero. Let B be a Borel subgroup and s be a semisimple element in B. Then $Z_G(s)^0 \cap B$ is a Borel subgroup of $Z_G(s)^0$.*

Proof. Let T be a maximal torus of G such that $s \in T \subset B$. Clearly $T \subset Z_G(s)^0$. Let U be the unipotent radical of B. Then $B = TU$. Note that $Z_G(s)^0 \cap B = T(U \cap Z_G(s)^0)$. As the characteristic of the underlying field is zero any unipotent group is connected and hence it follows that $Z_G(s)^0 \cap B$ is a connected group. Thus $Z_G(s)^0 \cap B$ is a connected solvable subgroup of $Z_G(s)^0$. We need to show that it is a maximal connected solvable group. Let B' be a Borel subgroup of $Z_G(s)^0$ containing $Z_G(s)^0 \cap B$. We will arrive at a contradiction by assuming $Z_G(s)^0 \cap B \neq B'$. Let U' be the unipotent radical of B'. Then $U \subset U'$ but $U \neq U'$. Consider the root system of G with respect to T and let Φ be the set of roots. Then the Borel subgroup B induces an ordering on Φ and the set of positive roots is denoted by Φ_+. Then $L(U) = \sum_{\alpha \in \Phi_+} L(G)_\alpha$. Consequently,

$$L\big(U \cap Z_G(s)^0\big) = \sum_{\alpha \in \Phi_+,\, \alpha(s)=1} L(G)_\alpha.$$

Since $Z_G(s)^0 \cap U \not\subset U'$ and both the groups are connected we conclude that $L(Z_G(s)^0 \cap U) \not\subset L(U')$. But this implies that there is an $\alpha_0 \in \Phi_+$ with $\alpha_0(s) = 1$ such that $L(G)_{-\alpha_0} \subset L(U')$. But then

$$L(G)_{-\alpha_0} + L(T) + L(G)_{\alpha_0} \subset L\big(B'\big).$$

This is a contradiction because $L(B')$ is a solvable Lie algebra but $L(G)_{-\alpha_0} + L(T) + L(G)_{\alpha_0}$ is not a solvable Lie algebra. This completes the proof. □

Corollary 5.2. *Let G be a Zariski connected reductive group over an algebraically closed field of characteristic zero. Let P be a parabolic subgroup and s be a semisimple element in P. Then $Z_G(s)^0 \cap P$ is a parabolic subgroup of $Z_G(s)^0$.*

Proof. Since $s \in P$ we may choose a Borel subgroup B of G so that $s \in B \subset P$. Now $Z_G(s)^0 \cap B \subset Z_G(s)^0 \cap P$. Note that by Lemma 5.1 it follows that $Z_G(s)^0 \cap B$ is a Borel subgroup of $Z_G(s)^0$. So $Z_G(s)^0 \cap P$ is a parabolic subgroup of $Z_G(s)^0$. □

Remark 5.3. Note that even if P is a proper parabolic subgroup the group $Z_G(s)^0 \cap P$ may not be a proper parabolic subgroup of $Z_G(s)^0$. We further remark that Lemma 5.1 and Corollary 5.2 can also be extended to any general Zariski connected algebraic group G.

In what follows some of the notions and results remain valid for groups over general fields, but for our purpose we will confine to the field of real numbers.

We need the following definitions from [15].

Definition 5.4. (See [15].) Let G be a connected complex reductive group defined over $\mathbb{R}$ and S maximal $\mathbb{R}$-split torus of G. Let $\Phi_{\mathbb{R}}$ be the set of $\mathbb{R}$-roots of G with respect to S and let $\Delta_{\mathbb{R}}$ be a set of simple roots in $\Phi_{\mathbb{R}}$. For $\theta \subset \Delta_{\mathbb{R}}$ define S_θ to be the identity component of $\bigcap_{\alpha\in\theta} \operatorname{Ker}(\alpha)$. The $\mathbb{R}$-split subtorus S_θ is called the *standard* $\mathbb{R}$*-split torus of type* θ. We say that a maximal $\mathbb{R}$-torus T is of *type* θ if the $\mathbb{R}$-split torus S_θ is the $\mathbb{R}$-split part of T, that is, $S_\theta = T_{sp}$.

We need the two following results from [15].

Theorem 5.5. *(See [15].) Let G be a Zariski connected complex reductive group defined over* $\mathbb{R}$*. Then any maximal* $\mathbb{R}$*-torus T of G is* $G(\mathbb{R})$*-conjugate to a maximal* $\mathbb{R}$*-torus of G having a standard* $\mathbb{R}$*-split torus as its* $\mathbb{R}$*-split part.*

See [15, Proposition 1.2] for a proof.

Retaining notations as in Definition 5.4, we denote by $W_{\mathbb{R}}$ the $\mathbb{R}$-*Weyl group* of G which is $W_{\mathbb{R}} = N_G(S)/Z_G(S)$.

Theorem 5.6. *(See [15].) Let G be a Zariski connected complex reductive group defined over* $\mathbb{R}$*. Let* T, T' *be two maximal* $\mathbb{R}$*-tori of G and* θ, θ' *be subsets of* $\Delta_{\mathbb{R}}$ *so that the maximal* $\mathbb{R}$*-split part T is* S_θ *and maximal* $\mathbb{R}$*-split part* T' *is* $S_{\theta'}$*. Then T is* $G(\mathbb{R})$*-conjugate to* T' *if and only if* θ *is* $W_{\mathbb{R}}$*-conjugate to* θ'*. Moreover all maximal* $\mathbb{R}$*-tori of G containing a maximal* $\mathbb{R}$*-split torus of G are* $G(\mathbb{R})$*-conjugate.*

See [15, Proposition 1.3 and Corollary 1.4] for a proof.

Recall that if G is a connected reductive group defined over $\mathbb{R}$ and $s \in G(\mathbb{R})$ is a semisimple element then $Z_G(s)^0$ is a connected reductive group defined over $\mathbb{R}$ containing s. The next corollary gives a description of maximal $\mathbb{R}$-split tori in $Z_G(s)^0$.

Corollary 5.7. *Let G be a Zariski connected complex reductive group defined over* $\mathbb{R}$*. Let s be a semisimple element in* $G(\mathbb{R})$*. Then any maximal* $\mathbb{R}$*-split torus of* $Z_G(s)^0$ *will be* $G(\mathbb{R})$*-conjugate to a standard* $\mathbb{R}$*-split torus* S_θ *(of type* θ*), for some* $\theta \subset \Delta_{\mathbb{R}}$*. Moreover, if* $\theta' \subset \Delta_{\mathbb{R}}$ *then some* $G(\mathbb{R})$*-conjugate of the standard* $\mathbb{R}$*-split torus* $S_{\theta'}$ *is a maximal* $\mathbb{R}$*-split torus of* $Z_G(s)^0$ *if and only if* θ *and* θ' *are* $W_{\mathbb{R}}$*-conjugate.*

Proof. As $s \in G(\mathbb{R})$, the group $Z_G(s)^0$ is defined over $\mathbb{R}$. Also note that as s is semisimple the rank of $Z_G(s)^0$ is the same as the rank of G. Now let $\overline{S}$ be a maximal $\mathbb{R}$-split torus of $Z_G(s)^0$. Then we can find a maximal $\mathbb{R}$-torus of $Z_G(s)^0$, say $\overline{T}$ containing $\overline{S}$. Clearly $\overline{T}$ is a maximal $\mathbb{R}$-torus of G. Moreover as $\overline{S} \subset \overline{T} \subset Z_G(s)^0$ and as $\overline{S}$ is a maximal $\mathbb{R}$-split torus of $Z_G(s)^0$ we conclude that $\overline{S}$ is the $\mathbb{R}$-split part of $\overline{T}$. Now we apply Theorem 5.5 to see that $\overline{T}$ is $G(\mathbb{R})$-conjugate to a maximal $\mathbb{R}$-torus having a standard $\mathbb{R}$-split torus, say S_θ, as its $\mathbb{R}$-split part. Hence $\overline{S} = aS_\theta a^{-1}$ for some $\theta \subset \Delta_{\mathbb{R}}$ and $a \in G(\mathbb{R})$.

Now suppose $\theta' \subset \Delta_{\mathbb{R}}$ and $b \in G(\mathbb{R})$ so that $bS_{\theta'}b^{-1}$ is a maximal $\mathbb{R}$-split torus of $Z_G(s)^0$. Let $S' = bS_{\theta'}b^{-1}$ and let T' be a maximal $\mathbb{R}$-torus of $Z_G(s)^0$ containing S'. As $\overline{T}$ and T are maximal $\mathbb{R}$-tori of $Z_G(s)^0$ containing the maximal $\mathbb{R}$-split tori (of $Z_G(s)^0$), $\overline{S}$ and S' respectively, by the last part of Theorem 5.6, we conclude that $\overline{T}$ and T' are $Z_G(s)^0(\mathbb{R})$ conjugate. Hence $a^{-1}Ta$ and $b^{-1}T'b$ are $G(\mathbb{R})$ conjugate. But S_θ is the $\mathbb{R}$-split part of $a^{-1}Ta$ and $S_{\theta'}$ is the $\mathbb{R}$-split part of $b^{-1}T'b$. Hence by Theorem 5.6 θ and θ' are $W_{\mathbb{R}}$ conjugate. Converse is clear as θ and θ' are $W_{\mathbb{R}}$ conjugate implies S_θ and $S_{\theta'}$ are $N_G(S)(\mathbb{R})$-conjugate. □

Definition 5.8. Recall that $W_{\mathbb{R}}$ is acting on $\Phi_{\mathbb{R}}$ and consider its corresponding action on the set of subsets of $\Phi_{\mathbb{R}}$. Two subsets σ_1, σ_2 of $\Phi_{\mathbb{R}}$ are said to be equivalent if there is a $w \in W_{\mathbb{R}}$ such that $w(\sigma_1) = \sigma_2$. The equivalence class of the subset σ of $\Phi_{\mathbb{R}}$ is denoted by $\bar{\sigma}$. Now we are in a position to define the type of an $\mathbb{R}$-rational semisimple element in a reductive group G defined over $\mathbb{R}$. Let $s \in G(\mathbb{R})$ be a semisimple element. Let S' be a maximal $\mathbb{R}$-split torus of $Z_G(s)^0$. Now by Corollary 5.7, S' is $G(\mathbb{R})$-conjugate to a standard $\mathbb{R}$-split torus S_θ, for some $\theta \subset \Delta_{\mathbb{R}}$. We define the *type* of s to be $\bar{\theta}$. Note that this is well defined because of Corollary 5.7. Moreover if $g \in G(\mathbb{R})$ then we define the *type* of g to be the type of g_s where g_s is the semisimple part of g. A subset θ of $\Delta_{\mathbb{R}}$ is said to be *admissible* if there is a maximal $\mathbb{R}$-torus T with S_θ being its $\mathbb{R}$-split part, that is, $S_\theta = T_{sp}$.

Remark 5.9. Note that type of g is $\overline{\Delta}_{\mathbb{R}}$ if and only if $Z_G(g_s)^0/Z(Z_G(g_s)^0)$ is $\mathbb{R}$-anisotropic if and only if $Z_G(g_s)^0/Z(Z_G(g_s)^0)(\mathbb{R})$ is a compact (connected group). This tells us that if type of g is $\overline{\Delta}_{\mathbb{R}}$ then the element g is semisimple. Also note that type of g is $\bar{\emptyset}$ (here $\emptyset$ denotes the empty set) if and only if $\mathbb{R}$-rank of $Z_G(g_s)^0$ is the same as the $\mathbb{R}$-rank of G.

Remark 5.10. Note that if T is a maximal $\mathbb{R}$-torus of G then there is an element $s \in T(\mathbb{R})$ so that $Z_G(s)^0 = T$. We have, $T(\mathbb{R})$ is Zariski dense in T and the set consisting of elements $x \in T$ for which $Z_G(x)^0 = T$ is a nonempty Zariski open subset in T. Thus there is an element $s \in T(\mathbb{R})$ so that $Z_G(s)^0 = T$. This says that $\theta \subset \Delta_{\mathbb{R}}$ is an admissible set if and only if there is an element $g \in G(\mathbb{R})$ of type $\bar{\theta}$. Further it can also be observed easily that $\theta \subset \Delta_{\mathbb{R}}$ is an admissible set if and only if the semisimple group $Z_G(S_\theta)/Z(Z_G(S_\theta))$ admits an $\mathbb{R}$-anisotropic maximal torus over $\mathbb{R}$ (see [15]).

Retain the notations introduced in Definition 5.8. We next prove Theorem 5.11 which tells us when an $\mathbb{R}$-rational element in $G(\mathbb{R})$ can be $G(\mathbb{R})$-conjugated to lie in a certain solvable $\mathbb{R}$-subgroup which in turn lies in a standard parabolic subgroup. Note that if $\theta, \theta' \subset \Delta_{\mathbb{R}}$ and if they are equivalent then θ is admissible if and only if θ' is so. Let $\mathcal{A}_G$ be a set of all *mutually nonequivalent admissible subsets* of $\Delta_{\mathbb{R}}$. Now for each θ in $\mathcal{A}_G$ we fix a maximal $\mathbb{R}$-torus, say T_θ, with S_θ as the $\mathbb{R}$-split part. Thus any maximal $\mathbb{R}$-torus of G is $G(\mathbb{R})$-conjugate to exactly one standard maximal torus T_θ of type θ, for some $\theta \in \mathcal{A}_G$. Also note that as S_θ is the $\mathbb{R}$-split part of T_θ we have, $T_\theta \subset Z_G(S_\theta) \subset P_\theta$, where P_θ is the standard $\mathbb{R}$-parabolic subgroup corresponding to θ (see Section 2).

Theorem 5.11. *Let G be a Zariski connected complex reductive group defined over $\mathbb{R}$ and $g \in G(\mathbb{R})$. Let g be of type $\bar{\theta}$ for some $\theta \in \mathcal{A}_G$. Let P_θ be the standard $\mathbb{R}$-parabolic subgroup corresponding to $\theta \subset \Delta_{\mathbb{R}}$. Then the following hold.*

(1) *Suppose that there is $\alpha \in G(\mathbb{R})$ so that $\alpha S_\theta \alpha^{-1}$ is a maximal $\mathbb{R}$-split torus of $Z_G(g_s)^0$. Then $Z_G(g_s)^0 \cap \alpha P_\theta \alpha^{-1}$ is a minimal $\mathbb{R}$-parabolic subgroup of $Z_G(g_s)^0$ with $Z_G(g_s)^0 \cap \alpha Z_G(S_\theta)\alpha^{-1}$ as a Levi part and $Z_G(g_s)^0 \cap \alpha R_u P_\theta \alpha^{-1}$ as the unipotent radical.*
(2) *There is a $\delta \in G(\mathbb{R})$ so that $\delta g_s \delta^{-1} \in T_\theta(\mathbb{R})$ and $\delta g_u \delta^{-1} \in R_u P_\theta(\mathbb{R})$.*

Consequently, if g is of type $\bar{\theta}$ then g can be $G(\mathbb{R})$-conjugated to lie in $T_\theta R_u P_\theta(\mathbb{R}) \subset P_\theta(\mathbb{R})$.

Proof. Let $s = g_s$ and $u = g_u$. As g is of type $\bar{\theta}$, a $G(\mathbb{R})$-conjugate of S_θ is a maximal $\mathbb{R}$-split torus of $Z_G(s)^0$.

First we prove part (1) of the theorem. Without loss of generality we may assume that $\alpha = e$. Thus S_θ is a maximal $\mathbb{R}$-split torus in $Z_G(s)^0$.

Claim 1. *Let $v \in Z_G(s)^0 \cap P_\theta(\mathbb{R})$ be a unipotent element. Then $v \in R_u P_\theta$.*

To prove the claim we first recall that

$$P_\theta = Z_G(S_\theta) R_u P_\theta.$$

Now as $S_\theta \subset Z_G(s)^0$, clearly $s \in Z_G(S_\theta)$. As $v \in P_\theta(\mathbb{R})$ it follows that $v = v_1 v_2$ where $v_1 \in Z_G(S_\theta)(\mathbb{R})$ and $v_2 \in R_u P_\theta(\mathbb{R})$. As v is unipotent we observe that v_1 is unipotent. Now as $s \in Z_G(S_\theta)$ and $v \in Z_G(s)^0$ we have

$$v_1 v_2 = v = svs^{-1} = sv_1 s^{-1} s v_2 s^{-1}.$$

As $sv_2 s^{-1} \in R_u P_\theta$ and $sv_1 s^{-1} \in Z_G(S_\theta)$ we conclude that $sv_1 s^{-1} = v_1$. Hence $v_1 \in Z_G(S_\theta) \cap Z_G(s)^0 = Z_{Z_G(s)^0}(S_\theta)$. Now as S_θ is a maximal $\mathbb{R}$-split torus in $Z_G(s)^0$ we conclude that $Z_{Z_G(s)^0}(S_\theta)/S_\theta$ has $\mathbb{R}$-rank zero. Thus all elements of $Z_{Z_G(s)^0}(S_\theta)(\mathbb{R})$ are semisimple. Note that v_1 is a unipotent element in $Z_{Z_G(s)^0}(S_\theta)(\mathbb{R})$ and hence we conclude that $v_1 = e$. So $v = v_2$ and hence $v \in R_u P_\theta$. This completes the proof of Claim 1.

As $s \in Z_G(S_\theta) \subset P_\theta$ it follows from Corollary 5.2 that $Z_G(s)^0 \cap P_\theta$ is a parabolic subgroup of $Z_G(s)^0$ which is defined over $\mathbb{R}$. We will show that $Z_G(s)^0 \cap P_\theta$ is a minimal $\mathbb{R}$-parabolic subgroup of $Z_G(s)^0$. Note that as S_θ is a maximal $\mathbb{R}$-split torus of $Z_G(s)^0$ and as $Z_G(s)^0 \cap P_\theta$ is a parabolic subgroup of $Z_G(s)^0$, it follows that there is a minimal $\mathbb{R}$-parabolic subgroup P' of $Z_G(s)^0$ satisfying

$$Z_{Z_G(s)^0}(S_\theta) \subset P' \subset Z_G(s)^0 \cap P_\theta.$$

Claim 2. $R_u P' \subset R_u P_\theta$.

As $R_u P'(\mathbb{R})$ is Zariski dense in $R_u P'$, to prove this claim it is enough to show that $R_u P'(\mathbb{R}) \subset R_u P_\theta$. Let $v \in R_u P'(\mathbb{R})$. Hence $v \in Z_G(s)^0 \cap P_\theta(\mathbb{R})$ is unipotent. Now by Claim 1 it follows that $v \in R_u P_\theta$. This completes the proof of Claim 2.

Since $Z_{Z_G(s)^0}(S_\theta) \subset P'$ and P' is a minimal $\mathbb{R}$-parabolic in $Z_G(s)^0$ it follows that $P' = Z_{Z_G(s)^0}(S_\theta) R_u P'$.

Claim 3. $Z_G(s)^0 \cap P_\theta = (Z_G(s)^0 \cap Z_G(S_\theta))(Z_G(s)^0 \cap R_u P_\theta)$.

To prove this claim we first want to show that

$$Z_G(s) \cap P_\theta = \big(Z_G(s) \cap Z_G(S_\theta)\big)\big(Z_G(s) \cap R_u P_\theta\big).$$

Let $x \in Z_G(s) \cap P_\theta$. Then $x = yz$ where $y \in Z_G(S_\theta)$ and $z \in R_u P_\theta$. Now $yz = x = sxs^{-1} = sys^{-1}szs^{-1}$, with $sys^{-1} \in Z_G(S_\theta)$ and $szs^{-1} \in R_u P_\theta$. Hence $y = sys^{-1}$, $z = szs^{-1}$. Thus $y \in Z_G(s) \cap Z_G(S_\theta)$ and $z \in Z_G(s)^0 \cap R_u P_\theta$. We now complete the proof Claim 3. Note that $R_u P_\theta$ is a unipotent group and hence $Z_G(s)^0 \cap R_u P_\theta = Z_G(s) \cap R_u P_\theta$. Now if $\bar{x} \in Z_G(s)^0 \cap P_\theta$ then, by the above observation, there exist $\bar{y} \in Z_G(s) \cap Z_G(S_\theta)$ and $\bar{z} \in Z_G(s) \cap R_u P_\theta$ such that $\bar{x} = \bar{y}\bar{z}$. Note that $\bar{z} \in Z_G(s)^0 \cap R_u P_\theta$ and hence $\bar{y} \in Z_G(s)^0 \cap Z_G(S_\theta)$. This completes the proof of Claim 3.

Now by Claim 3 it follows that

$$\begin{aligned} P' \subset Z_G(s)^0 \cap P_\theta &= \big(Z_G(s)^0 \cap Z_G(S_\theta)\big)\big(Z_G(s)^0 \cap R_u P_\theta\big) \\ &= Z_{Z_G(s)^0}(S_\theta)\big(Z_G(s)^0 \cap R_u P_\theta\big). \end{aligned}$$

Recall that by Claim 2 we have $R_u P' \subset Z_G(s)^0 \cap R_u P_\theta$. Since P' is a minimal $\mathbb{R}$-parabolic subgroup of $Z_G(s)^0$, the group $R_u P'$ is a maximal unipotent $\mathbb{R}$-subgroup of $Z_G(s)^0$. Hence $R_u P' = Z_G(s)^0 \cap R_u P_\theta$. Thus

$$P' = Z_{Z_G(s)^0}(S_\theta) R_u P' = Z_{Z_G(s)^0}(S_\theta)\big(Z_G(s)^0 \cap R_u P_\theta\big) = Z_G(s)^0 \cap P_\theta.$$

This completes the proof of (1). We now prove part (2) of the theorem. Again without loss of generality we may assume that S_θ is a maximal $\mathbb{R}$-split torus of $Z_G(s)^0$. Recall that if S_θ is a maximal $\mathbb{R}$-split torus of $Z_G(s)^0$ then by part (1) of the theorem it follows that $Z_G(s)^0 \cap P_\theta$ is a minimal parabolic subgroup of $Z_G(s)^0$ with $Z_{Z_G(s)^0}(S_\theta)$ as a Levi part and $Z_G(s)^0 \cap R_u P_\theta$ as the unipotent radical. Hence $Z_G(s)^0 \cap R_u P_\theta$ is a maximal $\mathbb{R}$-unipotent subgroup of $Z_G(s)^0$. Thus, as $u \in Z_G(s)^0(\mathbb{R})$ is unipotent, it follows that there is a $\beta \in Z_G(s)^0(\mathbb{R})$ such that $\beta u \beta^{-1} \in Z_G(s)^0 \cap R_u P_\theta$. Now as S_θ is a maximal $\mathbb{R}$-split torus of $Z_G(s)^0$ we can find a maximal $\mathbb{R}$-torus T' of $Z_G(s)^0$ such that $S_\theta \subset T'$. As s is in the center of the reductive group $Z_G(s)^0$ we have

$$s \in T' \subset Z_{Z_G(s)^0}(S_\theta).$$

Now note that both T_θ, T' are maximal $\mathbb{R}$-tori of the reductive group $Z(S_\theta)$ with a common $\mathbb{R}$-split part S_θ. So by Corollary 5.7 it follows that there is $\gamma \in Z(S_\theta)(\mathbb{R})$ such that $\gamma T' \gamma^{-1} = T_\theta$. As β commutes with s we have $(\gamma\beta)s(\gamma\beta)^{-1} = \gamma s \gamma^{-1} \in T_\theta$. Moreover as $Z(S_\theta)$ normalizes $R_u P_\theta$ it follows that $(\gamma\beta)u(\gamma\beta)^{-1} \in R_u P_\theta(\mathbb{R})$. We choose $\delta = \gamma\beta$. This completes the proof of (2). $\square$

Proof of Theorem 1.6. (1): Recall that H is a Levi subgroup of G which is defined over $\mathbb{R}$ and S is maximal $\mathbb{R}$-split torus of G contained in H. Let $g \in G(\mathbb{R})$. As a $G(\mathbb{R})$-conjugate of g_s lies in $H(\mathbb{R})$ we will assume, without loss of generality, that $g_s \in H(\mathbb{R})$. Let $g_u = u_1 u_2$ where $u_1 \in H(\mathbb{R})$ and $u_2 \in R_u G(\mathbb{R})$. Then clearly u_1 and u_2 are both unipotent elements and further, g_s commutes both with u_1 and u_2. Recall that $\Delta_{\mathbb{R}}^H$ is a set of simple roots in the root system of H with respect to the maximal $\mathbb{R}$-split torus S^H. Now choose $\theta \subset \Delta_{\mathbb{R}}^H$ such that g_s is of type θ in H. Then by (2) of Theorem 5.11, there is $\alpha \in H(\mathbb{R})$ so that $\alpha g_s \alpha^{-1} \in T_\theta(\mathbb{R})$

and $\alpha u_1 \alpha^{-1} \in R_u P_\theta^H(\mathbb{R})$. Thus $\alpha g_s \alpha^{-1} \in Z_{T_\theta(\mathbb{R})}(\alpha u_1 u_2 \alpha^{-1})$. Recall that $P_\theta = P_\theta^H R_u G$. Thus $R_u P_\theta = R_u P_\theta^H R_u G$ and $R_u P_\theta(\mathbb{R}) \subset \mathcal{U}_{G(\mathbb{R})}$. Clearly, $\alpha u_1 u_2 \alpha^{-1} \in R_u P_\theta(\mathbb{R})$. Thus there exists $X \in L(R_u P_\theta(\mathbb{R}))$ such that $Z_{T_\theta(\mathbb{R})}(\alpha u_1 u_2 \alpha^{-1}) = Z_{T_\theta(\mathbb{R})}(X)$. By the hypothesis of the statement it now follows that $\alpha g_s \alpha^{-1} \in P_n(Z_{T_\theta(\mathbb{R})}(\alpha g_u \alpha^{-1}))$. We now appeal to the equivalence of (1) and (2) of Theorem 3.3 to see that $\alpha g \alpha^{-1}$ has an n-th root in $G(\mathbb{R})$.

(2): First observe that any maximal torus T of G which is defined over $\mathbb{R}$ can be $G(\mathbb{R})$-conjugated to lie in H. Now appealing to Theorem 5.5, we see that T can be further $G(\mathbb{R})$ conjugated to T_θ for some admissible $\theta \subset \Delta_{\mathbb{R}}^H$. We now apply (1) of Theorem 1.6 to complete the proof.

(3): Let $g \in G(\mathbb{R})^*$. Then using the complete Jordan decomposition of g in $G(\mathbb{R})^*$ we get $g = g_e g_p$ (see Section 2). Now observe that $T_{an}(\mathbb{R})$ is a maximal (compact) torus of the maximal compact subgroup of $G(\mathbb{R})^*$. Hence there is an $\alpha \in G(\mathbb{R})^*$ such that $\alpha g_e \alpha^{-1} \in T_{an}(\mathbb{R})$. The proof now follows a path similar to that of (3) of Theorem 1.1 and hence is omitted. □

Remark 5.12. Theorem 1.6 gives only sufficient conditions for the surjectivity of $P_n : G(\mathbb{R}) \to G(\mathbb{R})$. However, the conditions are not necessary if n is even, as can easily be observed using Example 5.3 [14]. At present it is not clear to the author, for n odd, if conditions of Theorem 1.6 are also necessary for the surjectivity of $P_n : G(\mathbb{R}) \to G(\mathbb{R})$. However, in view of Theorem 1.1 we conclude that for a Zariski connected algebraic group G, the probable odd integers n for which conditions of Theorem 1.6 do not become necessary must come from the set of integers n for which $P_n : G \to G$ is not surjective.

6. Minimal parabolic subgroups, semiregular unipotent elements and power maps

In this section we prove Theorems 1.7, 1.8 and 1.9. We first need to recall some notations. The reader is referred to Section 21 [2] for generalities on absolute and relative roots. As in the previous sections, some of the notions and results remain valid for groups over general fields, but for our purpose we will confine to the field of real numbers $\mathbb{R}$. Let G be a Zariski connected reductive algebraic group over $\mathbb{R}$, S be a maximal $\mathbb{R}$-split torus of G and $\Phi_{\mathbb{R}}$ be the set of $\mathbb{R}$-roots with respect to S. Let P be a minimal $\mathbb{R}$-parabolic subgroup containing S. The set of positive roots induced from P and the set of simple roots thereof are denoted by ${\Phi_{\mathbb{R}}}^+$ and $\Delta_{\mathbb{R}}$, respectively. Let Φ_{nd} be the set of *non-divisible* roots (that is, roots α such that $\alpha/2$ is not a root) and let ${\Phi_{nd}}^+ = \Phi_{nd} \cap {\Phi_{\mathbb{R}}}^+$. For $\beta \in \Phi_{\mathbb{R}}$ we define $L(G)_\beta$ to be the root space in $L(G)$ corresponding to the root β. Now for $\alpha \in \Phi_{nd}$ we define $L(G)_{(\alpha)} = L(G)_\alpha$, if 2α is not a root and $L(G)_{(\alpha)} = L(G)_\alpha + L(G)_{2\alpha}$, if 2α is a root. For a root $\beta \in \Phi_{\mathbb{R}}$ we define $U_\beta = \exp(L(G)_\beta)$ and for $\alpha \in \Phi_{nd}$ we define $U_{(\alpha)} = \exp(L(G)_{(\alpha)})$. Note that for $\beta \in \Phi_{\mathbb{R}}$ the subset U_β need not be a group, but for $\alpha \in \Phi_{nd}$ the subset $U_{(\alpha)}$ is a unipotent group over $\mathbb{R}$; moreover $U_{(\alpha)} = U_\alpha$, if 2α is not a root and $U_{(\alpha)} = U_\alpha U_{2\alpha}$, if 2α is a root (in this case $U_{2\alpha}$ is the center of the group $U_{(\alpha)}$). By Theorem 21.9 [2] the groups $U_{(\alpha)}$ directly span $R_u P$, that is, $R_u P = \prod_{\alpha \in \Phi_{nd}^+} U_{(\alpha)}$. We need the following result from [1].

Theorem 6.1. *(See [1].) We retain the notations as in the above paragraph. Let $\Delta_{\mathbb{R}} = \{\alpha_1, \ldots, \alpha_n\}$ and $u \in R_u P(\mathbb{R})$ be of the form*

$$u = u_{\alpha_1} \cdots u_{\alpha_n} \cdot \prod_{\beta \in {\Phi_{\mathbb{R}}}^+ - \Delta_{\mathbb{R}}} u_\beta$$

where $u_{\alpha_i} \in U_{(\alpha_i)}(\mathbb{R})$, $u_\beta \in U_{(\beta)}(\mathbb{R})$. If $u_{\alpha_i} \neq e$ for all $i = 1, \ldots, n$ then $Z_G(u)(\mathbb{R}) \subset P$.

See Corollary 5 of [1] for a proof.

Proof of Theorem 1.7. In view of (2) of Theorem 1.3 it is enough to prove the theorem when G is a semisimple group over $\mathbb{R}$ which is of $\mathbb{R}$-rank one.

Let $P \subset G$ be a minimal parabolic subgroup defined over $\mathbb{R}$. Suppose that n is an odd integer and that $P_n : G(\mathbb{R}) \to G(\mathbb{R})$ is surjective. Let $g \in P(\mathbb{R})$ and let $S \subset P$ be a maximal $\mathbb{R}$-split torus of G. Let $\Phi_{\mathbb{R}}$ be the $\mathbb{R}$-roots of G with respect to S. We now impose the ordering on $\Phi_{\mathbb{R}}$ with respect to P. As G is of $\mathbb{R}$-rank one $\Delta_{\mathbb{R}}$ consists of one element, say α. Then $R_u P = U_{(\alpha)}$. Let us first assume that $g_u = e$. Then $g_s = g$. Now we can find an $\mathbb{R}$-torus $T \subset P$ such that $g_s \in T(\mathbb{R})$. As n is odd the map $P_n : T(\mathbb{R}) \to T(\mathbb{R})$ is surjective. Hence g has an n-th root in $T(\mathbb{R}) \subset P(\mathbb{R})$. Now suppose $g_u \neq e$. Then as $g_u \in P(\mathbb{R})$ it follows that $g_u \in U_{(\alpha)}$. Now as $P_n : G(\mathbb{R}) \to G(\mathbb{R})$ is surjective there is $h \in G(\mathbb{R})$ such that $g = h^n$. We will show that $h \in P(\mathbb{R})$. Note that h_s commutes with g_u and by Theorem 6.1 it follows that $h_s \in P(\mathbb{R})$. Moreover as $h_u^n = g_u$ it follows that the Zariski closure of the cyclic groups $\langle h_u \rangle$ and $\langle g_u \rangle$ are the same. Hence $h_u \in P(\mathbb{R})$. Thus $h \in P(\mathbb{R})$.

We now prove the converse. Now suppose that $P_n : P(\mathbb{R}) \to P(\mathbb{R})$ is surjective. Let $g \in G(\mathbb{R})$. As $\mathbb{R}$-rank of G is one it follows that either $Z_G(g_s)^0$ is $\mathbb{R}$-anisotropic or $\mathbb{R}$-rank of $Z_G(g_s)^0$ is one. In the first case as $Z_G(g_s)^0$ is $\mathbb{R}$-anisotropic the real Lie group $Z_G(g_s)^0(\mathbb{R})$ is compact and hence we have that $g_u = e$ (this is because $g_u \in Z_G(g_s)^0(\mathbb{R})$). Again as the group $Z_G(g_s)^0(\mathbb{R})$ is compact and connected we conclude that the exponential map $\exp : L(Z_G(g_s)^0(\mathbb{R})) \to Z_G(g_s)^0(\mathbb{R})$ is surjective and hence there is $h \in Z_G(g_s)^0$ so that $g_s = h^n$. In the second case when $\mathbb{R}$-rank of $Z_G(g_s)^0$ is one, by Theorem 5.11 the element g can be $G(\mathbb{R})$-conjugated to lie in $P(\mathbb{R})$. As $P_n : P(\mathbb{R}) \to P(\mathbb{R})$ is surjective it follows that we can find an n-th root of g in $G(\mathbb{R})$. $\square$

We now prove Theorem 1.8. Recall that a complex algebraic group G over $\mathbb{R}$ is $\mathbb{R}$-quasisplit if there is a Borel subgroup defined over $\mathbb{R}$. This is equivalent to saying that P is solvable where P is a minimal $\mathbb{R}$-parabolic subgroup of G. Further, a reductive group G over $\mathbb{R}$ is $\mathbb{R}$-quasisplit if and only if $Z_G(S) = T$, for some maximal torus T in G which is defined over $\mathbb{R}$. We need the following easy lemma.

Lemma 6.2. *Let K be a compact abelian Lie group, not necessarily connected, and $V \neq 0$ be a finite dimensional continuous $\mathbb{R}$-representation of K. Then there is a vector $v \in V$, $v \neq 0$ such that if an element $g \in K$ fixes v, that is, $gv = v$ then g fixes all the elements of V, that is, $gx = x$ for all $x \in V$.*

Proof. As the group K is compact we may assume that V is an irreducible $\mathbb{R}$-representation of K. In fact, as the group K is abelian, and V is irreducible, by Schur's Lemma, it is immediate that every vector $v \in V$, $v \neq 0$ satisfies the property stated in the lemma. $\square$

We need another lemma from [7] (a variant of this lemma also appears in [11]). We recall that an automorphism ψ of an algebraic group is said to be *semisimple* if its derivative $d\psi$ on the Lie algebra is a diagonalizable linear transformation.

Lemma 6.3. *Let U be a unipotent complex algebraic group and let ψ be a semisimple automorphism of U. If the automorphism of $U/[U, U]$ induced by ψ is the identity automorphism then so is ψ.*

Proof. See Corollary 2.5 of [7]. □

The proof of Theorem 1.8 is divided into two cases. In the first case it is shown that if G is reductive and $\mathbb{R}$-quasisplit then a semiregular unipotent element exists in $P(\mathbb{R})$, where P is a minimal $\mathbb{R}$-parabolic subgroup of G. This part can also be proved using properties of centralizers of regular nilpotent elements and Proposition 5.1 [23] where the analogous existence of regular nilpotent elements is obtained in $\mathbb{R}$-quasisplit reductive Lie algebras. However, our proof, in this case is new and more direct. In the latter case, when G is not necessarily reductive but $\mathbb{R}$-quasisplit, a technique akin to the proof of Proposition 3.1 [7] is applied to prove the existence of semiregular unipotent elements.

Proof of Theorem 1.8. *Case 1*: In this case we assume G to be reductive. Let P be a minimal $\mathbb{R}$-parabolic subgroup of G and let S be a maximal $\mathbb{R}$-split torus of G with $S \subset P$. Let us consider the $\mathbb{R}$-roots $\Phi_{\mathbb{R}}$ with respect to S and we impose the ordering on $\Phi_{\mathbb{R}}$ induced by P. Since G is $\mathbb{R}$-quasisplit $Z_G(S)$ will be a torus, say T, over $\mathbb{R}$. We retain the notations established in the beginning of the section. Let $\Delta_{\mathbb{R}} = \{\alpha_1, \ldots, \alpha_n\}$. Then

$$L\big(R_u P(\mathbb{R})\big) = L(G)_{\alpha_1}(\mathbb{R}) + \cdots + L(G)_{\alpha_n}(\mathbb{R}) + \sum_{\beta \in {\Phi_{\mathbb{R}}}^+ - \Delta_{\mathbb{R}}} L(G)_\beta(\mathbb{R}).$$

Claim 1. *There exist elements $v_i \in L(G)_{\alpha_i}(\mathbb{R})$, for each $i = 1, \ldots, n$, with $v_i \neq 0$ which have the following property. Let $w \in L(R_u P(\mathbb{R}))$ be any element of the form $w = v_1 + \cdots + v_n + z$, where $z \in \sum_{\beta \in {\Phi_{\mathbb{R}}}^+ - \Delta_{\mathbb{R}}} L(G)_\beta(\mathbb{R})$. If $g \in T(\mathbb{R})$ is any element satisfying $\mathrm{Ad}(g)w = w$ then $g \in Z(P)$.*

Let K be the maximal compact subgroup of $T(\mathbb{R})$. Then $T(\mathbb{R}) = KS(\mathbb{R})^*$ where this product is a direct product. Now note that for all $\alpha \in \Phi_{\mathbb{R}}$ the real vector space $L(G)_\alpha(\mathbb{R})$ is invariant under the adjoint action of K. We now use Lemma 6.2 to choose $v_i \in L(G)_{\alpha_i}(\mathbb{R})$, for each $i = 1, \ldots, n$, such that if $h \in K$ and $\mathrm{Ad}(h)v_i = v_i$ then $\mathrm{Ad}(h)x = x$ for all $x \in L(G)_{\alpha_i}(\mathbb{R})$. Now let $g \in T(\mathbb{R})$ and $z \in \sum_{\beta \in {\Phi_{\mathbb{R}}}^+ - \Delta_{\mathbb{R}}} L(G)_\beta(\mathbb{R})$ be such that $\mathrm{Ad}(g)(\sum v_i + z) = \sum v_i + z$. Then $\mathrm{Ad}(g)v_i = v_i$, for all $i = 1, \ldots, n$. Now $g = ks$ for some $k \in K$ and $s \in S(\mathbb{R})^*$. Note that as $\alpha_i(s)$ is positive real for every $i = 1, \ldots, n$ and as K is compact we have $\mathrm{Ad}(s)v_i = \mathrm{Ad}(k)v_i = v_i$, for all $i = 1, \ldots, n$. Thus $\mathrm{Ad}(s)x = x$, $\mathrm{Ad}(k)x = x$, for all $x \in \sum L(G)_{\alpha_i}(\mathbb{R})$. Now as $\sum L(G)_{\alpha_i}(\mathbb{R})$ generates $L(R_u P(\mathbb{R}))$ as a Lie algebra it follows that $\mathrm{Ad}(s)y = y$ and $\mathrm{Ad}(k)y = y$, for all $y \in L(R_u P(\mathbb{R}))$. Hence $\mathrm{Ad}(g)y = y$ for all $y \in L(R_u P(\mathbb{R}))$. By exponentiating we get that g commutes with all of $R_u P(\mathbb{R})$ and as $Z_G(S) = T$ is abelian it follows that $g \in Z(P)$.

Claim 2. *Keeping the notations as the above, if $g \in P(\mathbb{R})$ is a semisimple element with $\mathrm{Ad}(g)(\sum v_i) = \sum v_i$ then $g \in Z(P)$.*

Let $g \in P(\mathbb{R})$ be a semisimple element. Then there is a unipotent element $\bar{u} \in R_u P(\mathbb{R})$ such that $\bar{u} g \bar{u}^{-1} \in T(\mathbb{R})$. As $\mathrm{Ad}(g)(\sum v_i) = \sum v_i$ we have

$$\mathrm{Ad}\big(\bar{u} g \bar{u}^{-1}\big)\,\mathrm{Ad}(\bar{u})\Big(\sum v_i\Big) = \mathrm{Ad}(\bar{u})\Big(\sum v_i\Big).$$

Now $\mathrm{Ad}(\bar{u})(\sum v_i) = \sum v_i + z$ for some $z \in z \in \sum_{\beta \in {\Phi_k}^+ - \Delta_k} L(G)_\beta(\mathbb{R})$. So by Claim 1 it follows that $\bar{u} g \bar{u}^{-1} \in Z(P)$. Hence $g \in Z(P)$.

Now we choose $u = \exp(\sum v_i)$. As $v_i \neq 0$, for all $i = 1, \ldots, n$ it follows that the element u is an element of the form given in Theorem 6.1. Applying Theorem 6.1 we also see that $Z_G(u) = Z_P(u)$. This completes the proof of Case 1.

Case 2: In this case G is not necessarily reductive. Let H be a reductive Levi factor of G, defined over $\mathbb{R}$, such that $P \cap H$ is a minimal $\mathbb{R}$-parabolic subgroup of H. We denote $P \cap H$ by P_H. Clearly H is $\mathbb{R}$-quasisplit with P_H as a solvable minimal $\mathbb{R}$-parabolic subgroup.

We first assume that the unipotent radical R_uG is abelian. By Case 1, there is a unipotent element $u_1 \in P_H(\mathbb{R})$ which is semiregular in H. Let ρ denote the adjoint representation of $H(\mathbb{R})$ restricted to the Lie algebra $L(R_uG(\mathbb{R}))$. As H is a reductive group over $\mathbb{R}$, the representation ρ is completely reducible and let $L(R_uG(\mathbb{R})) = V_1 + \cdots + V_n$ be the decomposition of $L(R_uG(\mathbb{R}))$ into $H(\mathbb{R})$-irreducible (real) subspaces V_i, $i = 1, \ldots, n$. Let us denote the restriction of ρ on V_i by ρ_i. As u_1 is a unipotent element, the operators $\rho_i(u_1^{-1}) - 1$ are nilpotent operators on V_i and consequently we may choose $y_i \notin (\rho_i(u_1^{-1}) - 1)V_i$. Let $u_2 = \exp(\sum_{i=1}^n y_i)$. Clearly, $u_2 \in R_uG(\mathbb{R})$. We will now show that the unipotent element $u = u_1u_2 \in P(\mathbb{R})$ is semiregular in G. Let $g \in Z_G(u)(\mathbb{R})$ be a semisimple element. Then $g = tz$ for some $t \in H(\mathbb{R})$ and $z \in R_uG(\mathbb{R})$. Note that $z = \exp(\sum_{i=1}^n w_i)$ where $w_i \in V_i$. As g commutes with u it follows that t commutes with u_1 and consequently, by Case 1, $t \in Z(H)$. Further, as $tzu_1u_2 = u_1u_2tz$ and $t \in Z(H)$ we get

$$tu_1\bigl(u_1^{-1}zu_1u_2\bigr) = tu_1\bigl(t^{-1}u_2tz\bigr).$$

In other words $u_1^{-1}zu_1u_2 = t^{-1}u_2tz$. Thus we have

$$u_1^{-1}\exp\left(\sum_{i=1}^n w_i\right)u_1\exp\left(\sum_{i=1}^n y_i\right) = t^{-1}\exp\left(\sum_{i=1}^n y_i\right)t\exp\left(\sum_{i=1}^n w_i\right).$$

As R_uG is abelian and V_i are $H(\mathbb{R})$-invariant subspaces, from the above equation it follows that, for all i,

$$\rho_i\bigl(u_1^{-1}\bigr)w_i + y_i = \rho_i\bigl(t^{-1}\bigr)y_i + w_i.$$

In other words, $(\rho_i(t^{-1}) - 1)y_i = (\rho_i(u_1^{-1}) - 1)w_i$, for all i. But as $y_i \notin (\rho_i(u_1^{-1}) - 1)V_i$ we conclude that 1 is an eigenvalue of $\rho_i(t^{-1})$, for all i. Finally, as $t \in Z(H)$ and V_i are $H(\mathbb{R})$-irreducible, by Schur's Lemma, $\rho_i(t) = 1$, for all i. Thus $\rho(t) = 1$. As $\exp : L(R_uG(\mathbb{R})) \to R_uG(\mathbb{R})$ is surjective we conclude that t commutes with all of $R_uG(\mathbb{R})$. Now the Zariski density of $R_uG(\mathbb{R})$ in R_uG forces t to commute with all of R_uG. This implies that $t \in Z(G)$. As g is semisimple, appealing to the uniqueness of the Jordan decomposition, we get that $t = g$. Thus $g \in Z(G)$. This completes the proof of the case when R_uG is abelian.

We next consider the situation when R_uG is not necessarily abelian. Let $G' = G/[R_uG, R_uG]$ and $\pi : G \to G'$ be the natural projection map. Note that $P' = \pi(P)$ is a minimal parabolic subgroup of G' which is defined over $\mathbb{R}$ and R_uG' is abelian. Moreover, we note that $\pi(P(\mathbb{R})) = P'(\mathbb{R})$. By the above observation, there is a unipotent semiregular element $u' \in G'(\mathbb{R})$. We now fix a unipotent element $u \in P(\mathbb{R})$ so that $\pi(u) = u'$. We will show that u is a semiregular unipotent element in $G(\mathbb{R})$. Let $g \in Z_G(u)(\mathbb{R})$ be a semisimple element. Then $\pi(g) \in Z_{G'}(u')(\mathbb{R})$ is a semisimple element. Hence, by the case above, $\pi(g) \in Z(G')$. As $\pi(H)$ is a Levi part of G' and as $\pi(g) \in Z(G')$ it follows that $\pi(g) \in \pi(H)$. Observe that π is a bijection between H

and its image $\pi(H)$. Thus g can be assumed to be in $Z(H)$. Moreover, as the action of g on $R_uG/[R_uG, R_uG]$, induced by the conjugation by g on R_uG is trivial, by Lemma 6.3, g commutes with all of R_uG. Thus $g \in Z(G)$. This completes the proof. □

Lemma 6.4. *Let G be a Zariski connected complex algebraic group over $\mathbb{R}$ and P be a minimal $\mathbb{R}$-parabolic subgroup. If $s \in P(\mathbb{R})$ is a semisimple element then $Z_G(s)^0 \cap P$ is a minimal $\mathbb{R}$-parabolic subgroup of $Z_G(s)^0$.*

Proof. As $R_uG \subset P$, it follows that there exists a Levi subgroup $F \subset G$ defined over $\mathbb{R}$ such that $s \in F \cap P$ and $F \cap P$ is a minimal $\mathbb{R}$-parabolic subgroup of F. Clearly, as $s \in F$ and as $G = FR_uG$ we have $Z_G(s)^0 = Z_F(s)^0 Z_{R_uG}(s)$. Recall that as F is reductive so is $Z_F(s)^0$, which in turn forces $Z_F(s)^0$ to be a Levi subgroup of $Z_G(s)^0$ and $Z_{R_uG}(s)$ to be the unipotent radical of $Z_G(s)^0$. Let $P_F = F \cap P$. Then $P = P_F R_u G$ and $Z_G(s)^0 \cap P = (Z_F(s)^0 \cap P_F)Z_{R_uG}(s)$. Now as $s \in P_F(\mathbb{R})$ and P_F is a minimal $\mathbb{R}$-parabolic subgroup of F, by Theorem 5.11, $Z_F(s)^0 \cap P_F$ is a minimal $\mathbb{R}$-parabolic subgroup of $Z_F(s)^0$. Hence $(Z_F(s)^0 \cap P_F)Z_{R_uG}(s) = Z_G(s)^0 \cap P$ is a minimal $\mathbb{R}$-parabolic $Z_G(s)^0$. □

Proof of Theorem 1.9. (1): In view of Lemma 4.1, it is enough to prove that there is an $r \in G(\mathbb{R})$ with the property that $s = r^n$ and $r \in Z(Z_G(s)^0)$. As $Z_G(s)^0$ is $\mathbb{R}$-quasisplit, by Theorem 1.8, there is a unipotent element $u \in Z_G(s)^0(\mathbb{R})$ which is semiregular in $Z_G(s)^0$. Now consider the element $g \in G(\mathbb{R})$ given by $g = su$. Since $P_n : G(\mathbb{R}) \to G(\mathbb{R})$ is surjective, there is an $h \in G(\mathbb{R})$ with $g = h^n$. Clearly, $h_s^n = s$, $h_u^n = u$. We also observe that $h \in Z_G(s)^0(\mathbb{R})$. We further note that h_s commutes with u and hence $h_s \in Z_{Z_G(s)^0}(u)$. But as u is semiregular in $Z_G(s)^0$ we conclude that $h_s \in Z(Z_G(s)^0)$. We set $r = h_s$.

(2): Let now G be an $\mathbb{R}$-quasisplit group and P be a minimal $\mathbb{R}$-parabolic subgroup. Let $g \in P(\mathbb{R})$. Then by Lemma 6.4 it follows that $Z_G(g_s)^0 \cap P$ is a minimal $\mathbb{R}$-parabolic subgroup of $Z_G(g_s)^0$. But the group $Z_G(s)^0 \cap P$ is solvable, hence $Z_G(g_s)^0$ is a quasisplit group with $Z_G(g_s)^0 \cap P$ as a minimal $\mathbb{R}$-parabolic subgroup. Using part (1) of the theorem, we observe that there is an $r \in G(\mathbb{R})$ such that $r \in Z(Z_G(g_s)^0)$ and $g_s = r^n$. As $Z_G(g_s)^0 \cap P$ is a minimal $\mathbb{R}$-parabolic subgroup of $Z_G(g_s)^0$ it follows that $r \in Z_G(g_s)^0 \cap P$. Thus $r \in P_n(Z_{P(\mathbb{R})}(g_u))$. Thus, by Theorem 3.3, $g \in P_n(P(\mathbb{R}))$.

We next prove that $P_n : Z(G(\mathbb{R})) \to Z(G(\mathbb{R}))$ is surjective. Let $g \in Z(G(\mathbb{R}))$. As $g_s \in Z(G(\mathbb{R}))$ it follows that $Z_G(g_s)^0 = G$ and $Z_G(g_s)^0$ is $\mathbb{R}$-quasisplit. So we use part (1) as above to get $r \in G(\mathbb{R})$ so that $g_s = r^n$ and $r \in Z(Z_G(g_s)^0) = Z(G)$. Similarly, using Theorem 3.3, g has an n-th root in $Z(G(\mathbb{R}))$. This completes the proof of the theorem. □

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