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SECOND COHOMOLOGY OF COMPACT HOMOGENEOUS SPACES

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In Theorem 3.3 of [I. Biswas and P. Chatterjee, On the exactness of Kostant–Kirillov form and the second cohomology of nilpotent orbits, *Int. J. Math.* **23**(8) (2012)], the second cohomology of a quotient of a compact semisimple real Lie group was computed. In this addendum to the above paper, we give a simple topological proof of this theorem.

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1. Introduction

Let M be a compact connected semisimple real Lie group, and let $K \subset M$ be a closed subgroup. The connected component of K containing the identity element will be denoted by K^0. The center of the Lie algebra of K will be denoted by $\mathfrak{z}(\mathfrak{k})$. The adjoint action of K on $\mathfrak{z}(\mathfrak{k})$ clearly factors through the quotient K/K^0. Consider the action of K/K^0 on the dual vector space $\mathfrak{z}(\mathfrak{k})^*$ corresponding to the adjoint action of K/K^0 on $\mathfrak{z}(\mathfrak{k})$. Let

$$[\mathfrak{z}(\mathfrak{k})^*]^{K/K^0} \subset \mathfrak{z}(\mathfrak{k})^*$$

be the space of invariants for this action.

The following theorem was proved in [1] (see [1, Theorem 3.3]).

Theorem 1.1. *The cohomology $H^2(M/K, \mathbb{R})$ is canonically isomorphic to $[\mathfrak{z}(\mathfrak{k})^*]^{K/K^0}$.*

Our aim in this addendum to [1] is to give a simple topological proof of Theorem 1.1.

A geometric interpretation of the isomorphism in Theorem 1.1 is described in Remark 2.1.

2. Proof of Theorem 1.1

We can reduce to the case where M is simply connected by the following argument.

Let $\widetilde{M}$ be the universal cover on M; it is compact because M being semisimple its fundamental group is a finite one. The kernel of the projection

$$\alpha : \widetilde{M} \to M$$

lies in the center of $\widetilde{M}$. Let $\widetilde{K}_0$ be the connected component of $\widetilde{K} := \alpha^{-1}(K)$ containing the identity element. Since kernel(α) is contained in the center of $\widetilde{M}$, we have

$$[\mathfrak{z}(\mathfrak{k})^*]^{K/K^0} = [\mathfrak{z}(\mathfrak{k})^*]^{\widetilde{K}/\widetilde{K}_0}.$$

Also, $M/K = \widetilde{M}/\widetilde{K}$. Therefore, it is enough to prove Theorem 1.1 after substituting $(\widetilde{M}, \widetilde{K})$ for (M, K).

Hence we will assume that M is simply connected.

Consider the long exact sequence of homotopy groups for the quotient map

$$f : M \to M/K^0.$$

Since M is simply connected and K^0 is connected, we have $\pi_1(M/K^0) = 0$. As $\pi_2(M) = 0 = \pi_1(M)$, the same long exact sequence gives that $\pi_2(M/K^0) = \pi_1(K^0)$. Therefore, by Hurewicz' theorem

$$H_2(M/K^0, \mathbb{R}) = H_1(K^0, \mathbb{R}).$$

This implies that

$$H^2(M/K^0, \mathbb{R}) = H_2(M/K^0, \mathbb{R})^* = H_1(K^0, \mathbb{R})^* = H^1(K^0, \mathbb{R}).$$

But $H^1(K^0, \mathbb{R}) = \mathfrak{z}(\mathfrak{k})^*$. Indeed, for any $\omega \in \mathfrak{z}(\mathfrak{k})^*$, the 1-form on K^0 obtained by translating $\omega \in T_e^* K^0$ (right and left translations coincide) is d-closed, and the resulting homomorphism

$$\mathfrak{z}(\mathfrak{k})^* \to H^1(K^0, \mathbb{R})$$

is an isomorphism. Consequently, we have

$$H^2(M/K^0, \mathbb{R}) = \mathfrak{z}(\mathfrak{k})^*. \tag{2.1}$$

Now, consider the natural projection

$$\phi : M/K^0 \to M/K.$$

It is a Galois covering map with Galois group K/K^0. Therefore,

$$H^2(M/K, \mathbb{R}) = H^2(M/K^0, \mathbb{R})^{\mathrm{Gal}(\phi)} \subset H^2(M/K^0, \mathbb{R}), \tag{2.2}$$

where $H^2(M/K^0, \mathbb{R})^{\mathrm{Gal}(\phi)}$ is the space of invariants for the natural action of $\mathrm{Gal}(\phi)$ on $H^2(M/K^0, \mathbb{R})$. The isomorphism in (2.1) takes the action of $\mathrm{Gal}(\phi)$ on

$H^2(M/K^0, \mathbb{R})$ to the adjoint action of K/K^0 on $\mathfrak{z}(\mathfrak{k})^*$. Therefore, from (2.2) we get an isomorphism

$$H^2(M/K, \mathbb{R}) = [\mathfrak{z}(\mathfrak{k})^*]^{K/K^0}. \tag{2.3}$$

Remark 2.1. Take any homomorphism $\chi : K \to \mathbb{C}^*$. The group K acts on $M \times \mathbb{C}$ as follows: the action of any $g \in K$ sends any $(m, \lambda) \in M \times \mathbb{C}$ to $(mg, \chi(g^{-1}) \cdot \lambda)$. The projection $(M \times \mathbb{C})/K \to M/K$ is a line bundle, which we will denote by L^χ. Now, $\chi \mapsto c_1(L^\chi)$ produces a homomorphism

$$\mathrm{Char}(K) \otimes_{\mathbb{Z}} \mathbb{R} \to H^2(M/K, \mathbb{R}),$$

where $\mathrm{Char}(K)$ is the group of characters of K. On the other hand,

$$\mathrm{Char}(K) \otimes_{\mathbb{Z}} \mathbb{R} = [\mathfrak{z}(\mathfrak{k})^*]^{K/K^0}.$$

The resulting homomorphism

$$[\mathfrak{z}(\mathfrak{k})^*]^{K/K^0} \to H^2(M/K, \mathbb{R})$$

coincides with the isomorphism constructed in (2.3).

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Reference

[1] I. Biswas and P. Chatterjee, On the exactness of Kostant–Kirillov form and the second cohomology of nilpotent orbits, *Int. J. Math.* **23**(8) (2012) article ID: 1250086.