Towards Constructing Optimal Strip Move Sequences

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Abstract. The Sorting by Strip Moves problem, SBSM, was introduced in [6] as a variant of the well-known Sorting by Transpositions problem. A restriction called Block Sorting was shown in [2] to be NP-hard. In this article, we improve upon the ideas used in [6] to obtain a combinatorial characterization of the optimal solutions of SBSM. Using this, we show that a strip move which results in a permutation of two or three fewer strips or which exchanges a pair of adjacent strips to merge them into a single strip necessarily reduces the strip move distance. We also establish that the strip move diameter for permutations of size n is $n-1$. Further, we exhibit an optimum-preserving equivalence between SBSM and the Common Substring Removals problem (CSR) – a natural combinatorial puzzle. As a consequence, we show that sorting a permutation via strip moves is as hard (or as easy) as sorting its inverse.

1 Introduction

The Sorting by Strip Moves problem, SBSM, was introduced in [6] as a variant of the well-known Sorting by Transpositions problem SBT [1, 4, 5, 7]. Let π be a string which is a permutation of a finite set, say $\{1, 2, \ldots, n\}$. A transposition is the operation of picking up a substring of π and placing it elsewhere in the string. The SBT problem asks for a shortest sequence of transpositions that can transform π into the identity permutation $\mathtt{id}_n = 1\ 2 \ldots n$. A strip move is a restriction of a transposition: the substring being moved must be a maximal substring of π which is also a substring of $\mathtt{id}_n$. Such maximal substrings are called strips; for example, in the permutation 5 2 3 4 1 6 on 6 elements, there are four strips $\boxed{5}$, $\boxed{234}$, $\boxed{1}$ and $\boxed{6}$. The Sorting by Strip Moves problem, SBSM asks for a shortest sequence of strip moves that can transform π to $\mathtt{id}_n$.

Analyzing and understanding strip moves combinatorially is worthwhile for two reasons. Firstly, strips are an important combinatorial object in undertanding transpositions. In particular, strip move sorting directly gives a factor 3 approximation for sorting by transpositions [6], and there are optimal transposition sequences which do not, at any stage, break existing strips [3]. Secondly,

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a restriction, sorting by canonical strip moves (move a strip to a place where it merges with another strip) is shown in [2] to be NP-complete, while [6] shows that it is actually equivalent to SBSM. While [2] mentions various heuristic approaches to solving SBSM, and provides sufficient conditions under which a sorting sequence is optimal, [6] devises a 2-approximation algorithm for SBSM using certain combinatorial properites about optimal strip move sequences.

In this paper, we continue the task of exploring the combinatorial structure of optimal strip move sequences. We show that the strip move distance $s(\pi)$ of a permutation π equals the size of a largest *compatible edge set* in an associated *order graph* G_π (Theorem 1). This allows us to establish the strip-move diameter (Corollary 2) for permutations of size n; note that the corresponding question for general transpositions is still unsettled. Using our characterization, we establish that certain strip moves are optimal: they necessarily reduce the strip move distance. In particular, a strip move that reduces the number of strips in the permutation by 2 or 3 is optimal (Theorem 2). Also, if two adjacent strips are consecutive but out of order, then the strip move that exchanges them is optimal (Theorem 3). Thus, for instance, $s(\boxed{7\ 8}\,\boxed{4}\,\boxed{6}\,\boxed{5}\,\boxed{1\ 2\ 3}\,\boxed{9}) > s(\boxed{7\ 8}\,\boxed{4\ 5\ 6}\,\boxed{1\ 2\ 3}\,\boxed{9}) > s(\boxed{7\ 8}\,\boxed{1\ 2\ 3\ 4\ 5\ 6}\,\boxed{9})$.

We also establish the equivalence of Sorting By Strip Moves and the following combinatorial puzzle. Let π, ϕ be a pair of permutations of the same length, viewed as strings, with $\pi \neq \phi$. We wish to quantify how different these strings are. One way to do this is the following: identify a substring common to both π and ϕ and delete it. Repeat this process until both strings are the same. How many such deletions are needed? We denote the minimum number of deletions needed by $csr(\pi, \phi)$, and we define the *Common Substring Removals* problem, CSR, as the problem of computing $csr(\pi, \phi)$ for a given π, ϕ. For example, consider the pair $\pi = 3\ 5\ 1\ 4\ 2$ and $\phi = 1\ 4\ 3\ 2\ 5$ from S_5. An optimal common substring removal sequence that makes this pair identical is ($\mathbf{3}$ 5 1 4 2, 1 4 $\mathbf{3}$ 2 5) $\rightarrow$ (5 $\mathbf{1}$ $\mathbf{4}$ $\mathbf{2}$, $\mathbf{1}$ $\mathbf{4}$ $\mathbf{2}$ 5) $\rightarrow$ (5, 5). We show that SBSM and CSR are computationally equivalent (Theorem 4). The reductions preserve optimal values; thus the 2-approximation algorithm of [6] for SBSM is also a 2-approximation algorithm for the CSR problem. An interesting consequence of this equivalence is that sorting a permutation via strip moves is as hard (or as easy) as sorting its inverse (Corollary 3). The analogous result for transpositions is quite obvious, but, given the way strip moves are defined, there is no direct way of seeing this for strip move sorting.

Some of our results are subsumed by results of [2, 6], but the proof technique used here is completely different.

Due to space constraints, some proofs are omitted. See [8] for details.

2 Some preliminaries

We follow notation from [6]. We denote by $[n]$ the set $\{1, 2, \ldots, n\}$, by S_n the set of all permutations over $[n]$, and by $\mathtt{id}_n$ the identity permutation on n elements.

For $\pi \in S_n$, a strip of π is a maximal common substring of π and $\mathtt{id}_n$. A *substrip* is a substring of a strip. If a strip α is the substring $u\ u+1\ \ldots\ u+t$ for some u, t, then the strip containing $u-1$ (or $u+t+1$) is said to be the *predecessor* (*successor*, respectively) of α. Moving α to the immediate right of $u-1$ or to the immediate left of $u+t+1$ is called a *canonical strip move*. (If $u = 1$ or $u+t = n$, then moving α to the extreme left or right repsectively is also a canonical strip move.) The *strip move distance* $s(\pi)$ is the minimum number of strip moves needed to sort π; the corresponding minimum if strip moves are required to be canonical is the *canonical strip move distance* $sc(\pi)$.

Given a permutation π, replace each strip x by any one representative element to obtain a sequence S. Now, replacing each $a \in S$ by its rank in S, we obtain a permutation on k elements, where k is the number of strips in π. This permutation is called the kernel of π, denoted $\mathtt{ker}(\pi)$.

It is known that $s(\pi) = sc(\pi) = s(\mathtt{ker}(\pi))$ [6] and that computing $sc(\pi)$ is NP-hard [2].

3 Compatible edge sets and optimal solutions of SBSM

Given a permutation $\pi \in S_n$, we define an order graph G_π associated with it. We then define a property, which we call compatibility, of subsets of edges of G_π. We establish that optimally sorting a permutation π by strip moves is equivalent to finding a largest compatible edge set in G_π.

Definition 1. *The order graph of a permutation $\pi \in S_n$ is the directed graph $G_\pi = (V_\pi, E_\pi)$ where $V_\pi = [n]$ and $E_\pi = \{(u,v) \mid (u < v) \wedge (\pi_u^{-1} < \pi_v^{-1})\} = \{(\pi_i, \pi_j) \mid (i < j) \wedge (\pi_i < \pi_j)\}$.*

Definition 2.
1. *Two edges (u,v) and (w,x) of E_π are said to* interleave by value *if $u \leq w < v \leq x$. They are said to* interleave by position *if $\pi_u^{-1} \leq \pi_w^{-1} < \pi_v^{-1} \leq \pi_x^{-1}$. They are said to interleave if they interleave either by value or by position (or by both).*
2. *An edge (u,v) is said to* contain *an edge (w,x) of E_π if $u < w < x < v$ (contain by value) or $\pi_u^{-1} < \pi_w^{-1} < \pi_x^{-1} < \pi_v^{-1}$ (contain by position).*
3. *For any $C \subseteq E_\pi$, the inclusion graph $G(C,\pi)$ corresponding to C is the directed graph (C, A) where $A = \{((u,v),(w,x)) \mid (u,v)$ contains $(w,x)\}$.*
4. *A set $C \subseteq E_\pi$ is said to be compatible if no two edges of C interleave and if $G(C,\pi)$ is acyclic.*
5. *c_π is the size of any compatible set of edges of maximum cardinality.*

This definition of compatible edge sets generalises that of *non-crossing sets*, introduced in [6] to eficiently solve a certain variant of SBSM exactly.

Note that if C is a compatible edge set, then the subgraph $([n], C)$ is a collection of vertex-disjoint directed paths. Also, if C is a compatible edge set and $C' \subseteq C$, then C' is also compatible.

The main result of this section is that for all $\pi \in S_n$, $n \geq 1$, $s(\pi) = n-1-c_\pi$. We need to establish several lemmas to finally prove this theorem.

We call (u, v) a *unit edge* if $v - u = \pi_v^{-1} - \pi_u^{-1} = 1$. Thus unit edges are exactly those edges of the order graph G_π which link the adjacent elements of a strip. The following is easy to see.

Lemma 1. *Every compatible set in E_π of size c_π contains all unit edges of G_π.*

Definition 3. *For a permutation π, let C be any compatible subset of E_π. A strip α of π is said to be* free *with respect to C if every edge of C has neither or both endpoints in α.*

Lemma 2. *For a permutation π, let C be a compatible subset of E_π of size c_π, and let α be a strip of π free with respect to C. Moving α gives a permutation σ with $c_\sigma \geq c_\pi$. Furthermore, if α is moved canonically (to predecessor or successor) to obtain σ, then $c_\sigma > c_\pi$.*

Proof. Since α is a free strip, C is a subset of E_σ as well. Clearly, it has no edges interleaving by value. No edges interleave by new position either, since even in the new position no edge has just one endpoint in α. And the graph $G(C, \sigma)$ is acyclic as well, since the edges other than those in α form no cycles, and the unit edges within α do not contain any other edge by value or by new position. So C is a compatible subset of E_σ. Hence $c_\sigma \geq |C| = c_\pi$.

Further, if α is moved canonically, then G_σ has an extra unit edge created by this move, which is not even in C. So by Lemma 1, $|C| < c_\sigma$. Thus $c_\sigma > c_\pi$. $\square$

Lemma 3. *If σ is obtained from π via a strip/substrip move, then $c_\sigma \geq c_\pi - 1$. Furthermore, if σ is obtained from π via a canonical strip move, then $c_\sigma \geq c_\pi$.*

Proof. Let C be a compatible set of edges in G_π of size c_π. By Lemma 1, it contains all the unit edges of G_π. We now show that there is a compatible edge set C' of σ with $|C'| \geq |C| - 1$.

Let α be the strip or substrip moved in order to obtain σ from π. Let m be the number of edges of C with one endpoint in α and one endpoint outside α. Clearly, $m \in \{0, 1, 2\}$. If $m = 0$, then α is a free strip with respect to C, and the result follows from Lemma 2. So now assume that $m > 0$.

Case 1: m=1. Let (u, v) be the single edge of C with an endpoint in α and an endpoint outside it. (If α is a proper substrip, then this is in fact a unit edge.) As argued in the proof of Lemma 2, $C' = C \setminus \{(u, v)\}$ is a compatible edge set in E_π and E_σ, showing that $c_\sigma \geq c_\pi - 1$.

Further, if α is moved canonically, then G_σ has an extra unit edge created by this move, which is not in C'. So by Lemma 1, $|C'| < c_\sigma$ and hence $c_\sigma \geq c_\pi$.

Case 2: m=2. Let α be the (sub)strip $v+1, v+2, \ldots, v+k$, with $\pi_{v+j}^{-1} = i + j$ for $j \in [k]$. Since C contains all the unit edges of G_π, the two edges straddling α must be of the form $(u, v+1)$ and $(v+k, w)$. Let $C' = C \setminus \{(u, v+1), (v+k, w)\}$, and $C'' = C' \cup \{(u, w)\}$. As argued above, C' is a compatible edge set in E_π and in E_σ. (In particular, $G(C', \sigma)$ is acyclic.) We now show that C'' is a compatible edge set of G_σ, proving that $c_\sigma \geq c_\pi - 1$. We only need to show that adding (u, w) keeps this set compatible.

C has a path $\rho = u, v+1, v+2, \ldots, v+k, w$ which is replaced in C'' by two paths: a single-edge path $\rho' = u, w$ and the path $\rho'' = v+1, v+2, \ldots, v+k$; all other edges of C and C'' are the same. Since no other edge of C can interleave with ρ by value, it is clear that no other edge of C'' can interleave with paths ρ' and ρ'' by value. Since the relative positions of all elements other than $v+1, \ldots, v+k$ is unchanged, no edge of C and C'' can interleave with ρ' by position either. So C'' has no interleaving edges.

We must now show that $G(C'', \sigma)$ is acyclic. Assume it is not; then any cycle θ in it must pass through the vertex (u, w), since we already know that $G(C', \sigma)$ is acyclic. Since the moved unit edges corresponding to α do not contain any edge by value or by position, none of the corresponding vertices can figure in θ. Let (x_1, y_1) be predecessor and (x_2, y_2) the successor of (u, w) in θ. Then (x_1, y_1) must contain all edges of ρ in E_π. And either $(u, v+1)$ or $(v+k, w)$, both edges of ρ, must contain (x_2, y_2) in E_π. The remaining edges of θ are present in $G(C, \pi)$ as well. Thus, corresponding to θ, there is a cycle θ' in $G(C, \pi)$ as well, contradicting the compatibility of C.

Further, if α is moved canonically, then G_σ has an extra unit edge created by this move, which is not in C''. So by Lemma 1, $|C''| < c_\sigma$ and hence $c_\sigma \geq c_\pi$. $\square$

Lemma 4. *If π is obtained from σ by a strip move, then $c_\pi \leq c_\sigma + 1$.*

Proof. Given σ, let π be obtained through a move of some strip α on σ. Then α is either a strip or a substrip in π, and so, moving it back to its original place is a strip or substrip move from π yielding σ. Now apply Lemma 3. $\square$

Lemma 5. *Given $\pi \neq \mathtt{id}_n$ and a compatible set $C \subseteq E_\pi$ of size c_π, we can find a canonical strip move ρ on π leading to a σ with $c_\sigma = c_\pi + 1$.*

Proof. By Lemmas 2 and 4, it suffices to show that there exists a strip α free with respect to C. Consider the inclusion graph $G(C, \pi)$; it is acyclic by definition. All unit edges of G_π have zero out-degree in $G(C, \pi)$. Let C' be the set of non-unit edges of C; C' is also a compatible set. If $C' = \emptyset$, then C consists only of unit edges of G_π. Therefore, all strips of π are free with respect to C. Otherwise, consider the subgraph G' of $G(C, \pi)$ induced by the vertex set C'. By heredity, G' is also acyclic. Choose any vertex $(u, v) \in C'$ of zero out-degree in G'; let $i = \pi_u^{-1}$, $j = \pi_v^{-1}$. Let A be the set $\{u+1, \ldots, v-1\} \cup \{\pi_{i+1}, \ldots, \pi_{j-1}\}$. Since (u, v) is not a unit edge, A is non-empty. Since (u, v) is of out-degree zero in G', it contains no non-unit edge of C. So the strips containing any element of A must all be free with respect to C. $\square$

We now prove the main theorem of this section.

Theorem 1. *For all $\pi \in S_n$, $n \geq 1$, $s(\pi) = n - 1 - c_\pi = sc(\pi)$.*

Proof. We will show that (i) $s(\pi) \geq n - 1 - c_\pi$, and (ii) $sc(\pi) \leq n - 1 - c_\pi$. Since $s(\pi) \leq sc(\pi)$, the result follows.

To prove (i), let $\rho_1, \rho_2, \ldots, \rho_k$ be a sequence of strip moves sorting π. Let $\pi^0 = \pi$, and π^i be the permutation obtained by applying strip move ρ_i to

π^{i-1}. Let c_i denote c_{π^i}. By Lemma 4, $c_i \leq c_{i-1} + 1$. Since $\pi^k = \mathtt{id}_n$, and since $c_{\mathtt{id}_n} = n-1$, we have $n-1 = c_k \leq c_{k-1}+1 \leq \ldots \leq c_1+(k-1) \leq c_0+k = c_\pi+k$, hence $k \geq n-1-c_\pi$. This holds for any strip move sorting sequence; in particular, it holds for an optimal sequence with $k = s(\pi)$. Hence $s(\pi) \geq n-1-c_\pi$.

To prove (ii), repeatedly apply Lemma 5 and note that $c_{\mathtt{id}_n} = n-1$. □

Corollary 1. $s(\pi) = s(\mathtt{ker}(\pi))$.

If $\pi \in S_n$ and $\pi \neq \mathtt{rev}_n = n\ n-1 \ldots 2\ 1$, then clearly $c_\pi \geq 1$, and hence $s(\pi) \leq n-2$. For $\pi = \mathtt{rev}_n$, $c_\pi = 0$, and so $s(\mathtt{rev}_n) = n-1$. Thus we have:

Corollary 2. $D(n) = max\{s(\pi) | \pi \in S_n\} = n-1$.

Lemma 3 and Theorem 1 imply that canonical strip moves do not increase the strip-move distance. In the next two sections, we identify some canonical strip moves which provably reduce it.

4 2-moves and 3-moves are provably optimal

In this section, we establish that strip moves which reduce the number of strips by 2 or more necessarily reduce the strip move distance. This result is proved in many stages.

Lemma 6. *If $\pi \in S_n$ has $\pi_i = u-1$ and $\pi_{i+1} = v$ for some i and some $u < v$, and if the elements $u, u+1, u+2, \ldots, v-1$ form a strip α of π, then the permutation σ obtained from π by moving α to between u and v has $s(\sigma) < s(\pi)$.*

Proof. From Corollary 1, $s(\pi) = s(\mathtt{ker}(\pi))$. In $\mathtt{ker}(\pi)$, α is a single element. So it suffices to prove the lemma when $|\alpha| = 1$, i.e. $v = u+1$ and $\alpha = u$.

Let π have $\pi_i = u-1$ and $\pi_{i+1} = u+1$. By Lemma 2 and Theorem 1, it suffices to prove that there is some compatible edge set of size c_π with respect to which the strip containing u is free.

Let C be any compatible subset of E_π of size c_π. If u is free with respect to C there is nothing to prove. Otherwise, without loss of generality, assume that $k = \pi_u^{-1} > i+1$; the situation where $k < i$ is symmetric. There are three cases.
Case 1: There is a single edge (u, w) in C incident on u. Then C has no edge of the form $(x, u+1)$, since such an edge interleaves by value with (u, w). If C does not have any edge $(u-1, x)$, then let $C' = C \setminus \{(u, w)\} \cup \{(u-1, u+1)\}$.Otherwise let $(u-1, x)$ be an edge in C and let y be the rightmost endpoint of the path in C containing $u+1$. Let $C' = C \setminus \{(u, w), (u-1, x)\} \cup \{(u-1, u+1), (y, x)\}$. It can be seen that C' is a compatible subset of E_π of size c_π. And u is free with respect to C'.
Case 2: There is a single edge (w, u) in C incident on u. Then $(u-1, u+1)$ cannot be in C. If C has no edge with right endpoint $u+1$, then the set $C' = C \setminus \{(w, u)\} \cup \{(u-1, u+1)\}$ is a compatible subset of E_π of size c_π, satisfying Case 1 above. Otherwise, let $(x, u+1)$ be an edge in C. To avoid interleaving by value with (w, u), we have $x < w$. So $(x, u+1)$ contains (w, u) by value.

If $\pi_w^{-1} < \pi_x^{-1}$, then (w, u) contains $(x, u+1)$ by position, creating a cycle in $G(C, \pi)$.So $\pi_w^{-1} > \pi_x^{-1}$. But if $\pi_w^{-1} < i$, then $(x, u+1)$ and (w, u) would interleave by position. So the relative ordering of these elements must be $x, u-1, u+1, w, u$. Now the set $C' = C \setminus \{(w, u), (x, u+1)\} \cup \{(x, u-1), (u-1, u+1)\}$ is a compatible subset of E_π of size c_π, and u is free with respect to C'.
Case 3: There are two edges touching u, (w, u) and (u, x). There can be no edge into $u+1$ in C since any such edge will interleave with (u, x) by value. There can be no edge out of $u-1$ in C since any such edge will interleave with (w, u) by value, unless $w = u-1$. If $w = u-1$, set $C' = C \setminus \{(u-1, u), (u, x)\} \cup \{(u-1, u+1), (y, x)\}$ where y is the rightmost endpoint of the path in C containing $u+1$. Otherwise set $C' = C \setminus \{(w, u), (u, x)\} \cup \{(u-1, u+1), (w, x)\}$. Either way, C' is a compatible subset of E_π of size c_π, and u is free with respect to C'. □

Lemma 7. *If a permutation $\pi \in S_n$ has $\pi_{i-1} = u-1$ and $\pi_j = u$ for some $u, i, j \geq i+1, u$, and if the elements $\pi_i, \pi_{i+1}, \pi_{i+2}, \ldots, \pi_{j-1}$ form a strip α of π, then the permutation σ obtained from π by moving α canonically has $s(\sigma) < s(\pi)$.*

Proof. Since $s(\pi) = s(\texttt{ker}(\pi))$, it suffices to prove the lemma when $|\alpha| = 1$.

Let π have $\pi_{i-1} = u-1$, $\pi_{i+1} = u$, and $\pi_i = v$. By Lemma 2 and Theorem 1, it suffices to prove that there is some compatible edge set of size c_π with respect to which the strip containing v is free.

Let C be a compatible edge set of G_π of size c_π. If v is free with respect to C, there is nothing to prove. If C has a single edge e touching v, then edge $(u-1, u)$ cannot be in C since it interleaves with e by position. The set $C' = C \setminus \{e\} \cup \{(u-1, u)\}$ is a compatible set of G_π of size c_π, and v is free with respect to it. If C has two edges (w, v) and (v, x) incident on v, then $C' = C \setminus \{(w, v), (v, x)\} \cup \{(w, x), (u-1, u)\}$ is a compatible set in G_π of size c_π, with respect to which v is free. □

Since the goal in SBSM is to move from an unsorted permutation towards $\texttt{id}_n$ which has just one strip, it is natural to prefer strip moves that rapidly reduce the number of strips. With the results we have seen, we can formalise this intuition. Define an x-move to be a strip move that results in a permutation with x fewer strips. (Note that a strip move cannot decrease the number of strips present by more than 3.) Since a 2-move satisfies the assumption of Lemmas 6 or 7 and since a 3-move satisfies the assumption of both, we now have the main result of this section:

Theorem 2. *If a 2-move or a 3-move applied to π gives σ, then $s(\sigma) < s(\pi)$.*

5 Certain 1-moves are provably optimal

We furnish another application of Theorem 1. We show that a strip move that exchanges a pair of adjacent substrings which are adjacent in the identity permutation can be part of an optimal strip move sorting sequence. Note that such a move may not even be a 2-move. To prove this, we establish an interesting property, which we call *additivity*, of the function $s(\pi)$.

Theorem 3. *If α, β are adjacent strips in π such that $\beta\alpha$ is a substring of $\texttt{id}_n$, then ϕ obtained from π by exchanging α and β (which is a canonical move of α or β) has $s(\phi) < s(\pi)$.*

Proof. Let the substring $\beta\alpha$ be $u+1\ u+2\ \ldots\ u+t$ for some u, t, and let it occur in positions $i+1,\ i+2,\ \ldots,\ i+t$ of ϕ for some i. If $\pi_i = u$ or if $\pi_{i+t+1} = u+t+1$, then exchanging α and β in π is a 2-move or a 3-move, and hence, by Theorem 2, reduces the strip move distance. So now let us assume that $\pi_i \neq u$ and $\pi_{i+t+1} \neq u+t+1$. Clearly, one way to sort π is to exchange α and β and then sort ϕ; thus $s(\pi) \leq 1 + s(\phi)$. We show that this is in fact an equality, by defining a notion of *complete* substrings and establishing a more general result concerning them. The proof follows from the statement of Lemma 8 below and the fact that $\alpha\beta$ forms a maximal complete substring in π. $\square$

Definition 4. *A substring $\pi_{i+1} \ldots \pi_{i+t}$ is* complete *if $\{\pi_{i+1}, \ldots, \pi_{i+t}\} = \{u+1, u+2, \ldots, u+t\}$ for some u. That is, the t consecutive positions $i+1, i+2, \ldots, i+t$ hold t consecutive integers, though not necessarily in sorted order. If, furthermore, $\pi_i \neq u$ and $\pi_{i+t+1} \neq u+t+1$, then the substring is said to be* maximal complete.

The strip move distance for the complete substring $\pi_{i+1} \ldots \pi_{i+t}$ is defined to be the minimum number of strip moves needed to sort it i.e. to transform it into $u+1 \ldots u+t$.

Lemma 8. *If π contains a maximal complete substring α with $s(\alpha) = q$, then $s(\pi) = q + s(\sigma)$, where σ is the permutation obtained from π by replacing α, in place, with its sorted version.*

Proof. Since α is maximal complete, it is clear that $s(\pi) \leq q + s(\sigma)$. So we only need to show that $s(\pi) \geq q + s(\sigma)$. By Theorem 1, it suffices to show that $c_\sigma \geq q + c_\pi$.

Let $\alpha = \pi_{i+1} \ldots \pi_{i+t}$ where $\{\pi_{i+1}, \ldots, \pi_{i+t}\} = [u+1,\ u+t]$ for some u. Let C be any compatible edge set of G_π of size c_π. Partition C into four subsets as follows:

$C_1 = \{(v,w) \mid v, w \in [u+1,\ u+t]\}$	both endpoints in $[u+1,\ u+t]$
$C_2 = \{(v,w) \mid \pi_v^{-1} \leq i, w \in [u+1,\ u+t]\}$	right endpoint in $[u+1,\ u+t]$
$C_3 = \{(v,w) \mid v \in [u+1,\ u+t], \pi_w^{-1} > i+t\}$	left endpoint in $[u+1,\ u+t]$
$C_4 = C \setminus (C_1 \cup C_2 \cup C_3)$	no endpoint in $[u+1,\ u+t]$

It follows from Theorem 1 that $|C_1| \leq c_\alpha = t-1-q$; i.e. $t-1-|C_1| \geq q$. If $|C_2| \leq 1$ and $|C_3| \leq 1$, we construct a compatible set in E_σ of the required size as follows: Let C_5 be the set of $t-1$ unit edges obtained after sorting α; $C_5 = \{(u+i, u+i+1) \mid i \in [t-1]\}$. Define $C' = C_4 \cup C_5$. Then C' is a subset of E_σ, and it is easy to see that C' is compatible. If $C_2 \cup C_3 = \emptyset$, then $|C_1| + |C_4| = c_\pi$, so C' is the desired set. Otherwise, Otherwise, $C'' = C' \cup \{(v, u+1) \mid (v, u+i) \in C_2 \text{ for some } i \in [t]\} \cup \{(u+t, w) \mid (u+j, w) \in C_3 \text{ for some } j \in [t]\}$ is the desired set; compatibility and size bounds are easily seen to hold.

If either $|C_2|$ or $|C_3|$ exceeds 1, we show how to construct another compatible subset of E_π of size c_π, in which the corresponding subsets $|C_2|$ and $|C_3|$ are both of size at most 1. Then we can use the above argument.

Assume that $|C_2| = a \geq 2$. Let the edges of C_2 be $(\pi_{i_1}, \pi_{j_1}), (\pi_{i_2}, \pi_{j_2}), \ldots, (\pi_{i_a}, \pi_{j_a})$, where $j_1 < j_2 < \ldots < j_a$. By compatibility considerations, we have $i_a < \ldots < i_2 < i_1 < j_1 < j_2 < \ldots < j_a$. Let the unique path in C containing π_{j_h} end at the right at π_{k_h}. From compatibility considerations, it can be argued that $(\pi_{k_h}, \pi_{j_{h+1}}) \in E_\pi$. We define $C'_2 = \{(\pi_{i_1}, \pi_{j_1})\} \cup \{(\pi_{k_h}, \pi_{j_{h+1}}) \mid h \in [a-1]\}$.

If $|C_3| = b \geq 2$, then a set C'_3 of the same size is constructed in a symmetric way. Let the edges of C_3 be $(\pi_{p_1}, \pi_{q_1}), (\pi_{p_2}, \pi_{q_2}), \ldots, (\pi_{p_r}, \pi_{p_r})$, where $p_b < \ldots < p_2 < p_1$. By compatibility considerations, we have $p_b < \ldots < p_2 < p_1 < q_1 < q_2 < \ldots < q_b$. Let the unique path in C containing π_{p_h} end at the left at π_{r_h}. Again, we can argue that the edges $(\pi_{p_{h+1}}, \pi_{r_h})$ are indeed present in E_π, and we define $C'_3 = \{(\pi_{p_1}, \pi_{q_1})\} \cup \{(\pi_{p_{h+1}}, \pi_{r_h}) \mid h \in [b-1]\}$;

If both $a, b \geq 2$, then it must be that $j_a < p_b$, since otherwise there will be edges in C interleaving by position.

Clearly, $C'_2 \cup C'_3$ is of the same size as $C_2 \cup C_3$, and is disjoint from $C_1 \cup C_4$. We now show that $C' = C_1 \cup C'_2 \cup C'_3 \cup C_4$ is also a compatible subset of E_π.

It is easy to see that none of the edges of $C'_2 \cup C'_3$ interleave by position. Neither do edges of $C_1 \cup C_4$. And by our choice of C', an edge of $C'_2 \cup C'_3$ will not interleave by position with an edge of $C_1 \cup C_4$.

If edges of C' interleave by value, then the corresponding edges of C can similarly be shown to interleave by value, since α is a complete substring.

It can be argued that if the inclusion graph $G(C', \pi)$ has a cycle, then so does $G(C, \pi)$, contradicting compatibility of C. $\square$

Note: The above result can be shown to hold even if α is complete but not maximal.

6 CSR and SBSM are equivalent

We first obtain an alternative formulation of SBSM and then show that this alternative formulation is equivalent to CSR. This then implies that sorting a permutation via strip moves is as hard (or as easy) as sorting its inverse.

We say that an increasing substring $a_i \ldots a_j$ of a string $S = a_1\ a_2 \ldots a_m$ of distinct elements from $[n]$ is *tight* if there is no $i \leq k \leq j-1$ such that $a_k < a_l < a_{k+1}$ for some a_l in S. (In other words, if each a_l in S is replaced by its rank $r(a_l)$ in S, then the substring $r(a_i) \ldots r(a_j)$ is a substrip in the resulting string.) Let $isr(S)$ denote the length of a shortest sequence of tight increasing substring removals on S that leaves behind an increasing subsequence of S.

For example, consider 3 5 1 4 2. Neither of the increasing substrings $\langle 3, 5\rangle$ and $\langle 1, 4\rangle$ is tight, and removing any one element does not make the sequence increasing, so $isr(3\ 5\ 1\ 4\ 2) > 1$. But after deleting 4, the sequence $\langle 3, 5\rangle$ is tight, and deleting it leaves behind an increasing sequence 1 2. So $isr(3\ 5\ 1\ 4\ 2) = 2$.

The following is easy to see.

Lemma 9. *For any permutation* π*,* $isr(\pi) = s(\pi)$*.*

Recall that $csr(\pi, \phi)$ denotes the length of a shortest sequence of common substring removals that makes the pair identical. Given any $\pi, \phi \in S_n$, for any permutation $\psi \in S_n$, $(\psi^{-1}\pi, \psi^{-1}\phi)$ is just a consistent re-labelling for (π, ϕ). Thus $csr(\pi, \phi) = csr(\psi^{-1}\pi, \psi^{-1}\phi)$. In particular, $csr(\pi, \phi) = csr(\pi^{-1}\phi, \mathtt{id}_n) = csr(\phi^{-1}\pi, \mathtt{id}_n)$. But if we consider $csr(\sigma, \mathtt{id}_n)$, the common substrings removed are always tight increasing substrings. Thus $csr(\sigma, \mathtt{id}_n) \geq isr(\sigma)$. In fact, we have equality $csr(\sigma, \mathtt{id}_n) = isr(\sigma)$ because the tight increasing substrings removed from σ are necessarily substrings of $\mathtt{id}_n$. These remarks, along with Lemma 9, establish the following theorem.

Theorem 4. *The Sorting by Strip Moves problem SBSM and the Common Substring Removals problem CSR are computationally equivalent.*
For $\pi, \phi \in S_n$*,* $csr(\pi, \phi) = s(\pi^{-1}\phi) = s(\phi^{-1}\pi)$*. For* $\sigma \in S_n$*,* $s(\sigma) = csr(\sigma, \mathtt{id}_n)$*.*

Corollary 3. *For any* $\pi \in S_n$*,* $s(\pi) = s(\pi^{-1})$*.*

7 Conclusion

In this article, we have shown that certain types of strip moves are optimal; they necessarily reduce the strip move distance. Analogous results for sorting by transpositions would be very useful. In particular, identifying certain transpositions that provably reduce the transposition distance could guide the way to efficiently sorting by transpositions, either optimally or with good approximation ratios. We believe that the techniques used in this paper can find application to the general sorting by transpositions problem as well.

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