# Lower bounds for linear decision lists

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**Abstract**

We demonstrate a lower bound technique for linear decision lists, which are decision lists where the queries are arbitrary linear threshold functions. We use this technique to prove an explicit lower bound by showing that any linear decision list computing the function $\mathsf{MAJ} \circ \mathsf{XOR}$ requires size $2^{0.18n}$. This completely answers an open question of Turán and Vatan [18]. We also show that the spectral classes $\mathsf{PL}_1, \mathsf{PL}_\infty$, and the polynomial threshold function classes $\widehat{\mathsf{PT}}_1, \mathsf{PT}_1$, are incomparable to linear decision lists.

## 1 Introduction

Decision lists are a widely studied model of computation, first introduced by Rivest [17]. A decision list $L$ of size $\ell$ computing a Boolean function $f \in B_n$ is a sequence of $\ell - 1$ instructions of the form **if** $f_i(x) = a_i$ **then output**

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[*]This work was done while the author was a graduate student at TIFR, Mumbai.

$b_i$ **and stop**, followed by the instruction **output** $\neg b_{\ell-1}$ **and stop**. Here $B_n$ denotes the set of all Boolean functions in $n$ variables, each $f_i \in B_n$ is called a *query function*, and $a_i$ and $b_i$ are Boolean constants. If the functions $f_i$ all belong to a function class $S \subseteq B_n$, then $L$ is said to be an $S$-decision list.

Krause [14] showed that there are functions with small representation as AND-decision lists, but requiring exponential size when computed by depth-two circuits with a linear threshold gate at the top and XOR gats at the bottom. On the other hand, Impagliazzo and Williams [13] showed that a certain condition is sufficient to prove lower bounds against a related computation model that can be termed rectangle-decision lists. Linear decision lists are decision lists where the query functions are linear threshold functions. Lower bounds against linear decision lists (and even against bounded-rank linear decision trees, a natural generalisation) for the Inner Product modulo 2 function were proved by Gröger, Turán and Vatan, in [9, 18]. Subsequently, Uchizawa and Takimoto [19, 20] showed lower bounds against the class of linear decision lists and linear decision trees when the weights of the linear threshold queries are bounded by a polynomial in the input length. In fact, the lower bounds of [19, 20] apply to any function with large *unbounded-error communication complexity*.

We observe that the lower bound argument in [18] shows that functions efficiently computable by linear decision lists (with no restrictions on the weights of the queried linear threshold functions) must have large monochromatic rectangles. In fact, we build on their argument to establish a more general result (Lemma 17). Informally, we show that if a function has no "large" weight monochromatic rectangles under some product distribution then it cannot be expressed by "small" linear decision lists. We then use this fact to establish a lower bound for a seemingly simple function, MAJ ∘ XOR (see Definition 18). Our main theorem is as follows.

**Theorem 1.** *Any linear decision list computing* $\mathsf{MAJ}_n \circ \mathsf{XOR}$ *must have size* $2^{\Omega(n)}$.

It is not hard to see that MAJ ∘ XOR can be simulated by MAJ ∘ MAJ circuits with only a *linear* blow-up in size. This immediately yields the following corollary, resolving an open question posed by Turán and Vatan in [18].

**Corollary 2.** *There exists a function that can be computed by polynomial sized* MAJ ∘ MAJ *circuits, but any linear decision list computing it requires exponential size.*

Impagliazzo and Williams [13] demonstrated a function, implicitly computable by polynomial sized MAJ ∘ MAJ circuits, which cannot be computed

by polynomial sized rectangle-decision lists. We observe that our lower bound technique against linear decision lists (Lemma 17) coincides with the sufficient condition considered in [13] to prove lower bounds against rectangle-decision lists. Thus, their function also separates linear decision lists from $\mathsf{MAJ} \circ \mathsf{MAJ}$. However, we obtain a $2^{\Omega(n)}$ lower bound on the length of linear decision lists in Theorem 1, improving upon the bound implicit in the work of Impagliazzo and Williams, which is worse in the exponent by at least a quadratic factor. Very recently, Chattopadhyay, Mande and Sherif [5] showed several properties of the function $\mathsf{SINK} \circ \mathsf{XOR}$. We observe that as a consequence, our lower bound technique against linear decision lists (Lemma 17) also applies to this function. We elaborate more on these remarks in Section 5.

## 2 Preliminaries

**Definition 3** (Sign function). *The function* $\mathsf{sign} : \mathbb{R} \to \{0, 1\}$ *is defined as follows.*

$$\mathsf{sign}(x) = \left\{ \begin{array}{ll} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{array} \right.$$

**Definition 4** (Linear Threshold Functions). *A function* $f : \{0, 1\}^n \to \{0, 1\}$ *is said to be a linear threshold function (*$\mathsf{LTF}$*) if there exist real numbers* $w_0, w_1, \ldots, w_n$ *such that* $f(x) = \mathsf{sign} \left( w_0 + \sum\limits_{i=1}^{n} w_i x_i \right)$.

For strings $x, y \in \mathbb{R}^n$, we denote their inner product by $\langle x, y \rangle \triangleq \sum_i x_i y_i$. With this notation, $f$ is an $\mathsf{LTF}$ if for some $w_0 \in \mathbb{R}$, $\tilde{w} \in \mathbb{R}^n$, $f(x) = \mathsf{sign}(w_0 + \langle \tilde{w}, x \rangle)$.

**Definition 5** (Majority). *The function* $\mathsf{MAJ}_n : \{0, 1\}^n \to \{0, 1\}$ *is the linear threshold function defined by* $\mathsf{MAJ}_n(x) = \mathsf{sign}(x_1 + x_2 + \cdots + x_n - n/2)$.

**Definition 6** (Function composition). *For functions* $f : \{0, 1\}^n \to \{0, 1\}$ *and* $g : \{0, 1\}^m \to \{0, 1\}$*, the function* $f \circ g : \{0, 1\}^{nm} \to \{0, 1\}$ *is defined as follows:*

$$f \circ g(x_{11}, \ldots, x_{1m}, \ldots, x_{n1}, \ldots, x_{nm}) = f(g(x_{11}, \ldots, x_{1m}), \cdots, g(x_{n1}, \ldots, x_{nm})).$$

We now formally define the model of computation that is of interest in this paper.

**Definition 7** (Linear Decision Lists). *A linear decision list (*LDL*) of size $k$ is a sequence $(L_1, a_1), (L_2, a_2), \ldots, (L_k, a_k)$, where each $a_i \in \{0, 1\}$, and each $L_i$ is an* LTF *with $L_k$ being the constant function 1. The decision list computes a function $f : \{0, 1\}^n \to \{0, 1\}$ as follows : If $L_1(x) = 1$, then $f(x) = a_1$; elseif $L_2(x) = 1$, then $f(x) = a_2$; elseif . . . , elseif $L_k(x) = 1$, then $f(x) = a_k$. That is,*

$$f(x) = \bigvee_{i=1}^{k} \left( a_i \wedge L_i(x) \wedge \bigwedge_{j<i} \neg L_j(x) \right).$$

**Definition 8** (Communication matrix). *For a function $F : \{0, 1\}^n \times \{0, 1\}^n \to \{0, 1\}$, its communication matrix $M_F$ is the $2^n \times 2^n$ matrix with entries $M_F[x, y] := F(x, y)$.*

**Definition 9** (Monochromatic rectangles/squares). *Let $F : \{0, 1\}^n \times \{0, 1\}^n \to \{0, 1\}$ be any function. For $b \in \{0, 1\}$, a monochromatic $b$-rectangle is a tuple $(X, Y)$, where $X, Y \subseteq \{0, 1\}^n$ and $F(x, y) = b$ for every $(x, y) \in X \times Y$. We say that $(X, Y)$ is a monochromatic square of size $s$ if it is a monochromatic 0-rectangle or 1-rectangle and, furthermore, $|X| = |Y| = s$.*

**Definition 10** (Product distributions and weights). *A probability distribution $\eta$ over $\{0, 1\}^n \times \{0, 1\}^n$ is said to be a* product distribution *if there are probability distributions $\mu$, $\nu$ over $\{0, 1\}^n$ such that for every $(x, y) \in \{0, 1\}^n \times \{0, 1\}^n$, $\eta(x, y) = \mu(x) \times \nu(y)$. We say that $\eta$ is the product distribution $\mu \times \nu$.*

*Given a probability distribution $\mu$ over $\{0, 1\}^n$ and $X \subseteq \{0, 1\}^n$, $\mu(X)$ is defined to be the sum $\sum_{x \in X} \mu(x)$.*

*For a rectangle $(X, Y)$, its* weight *under a product distribution $\mu \times \nu$ is $(\mu \times \nu)(X \times Y) = \mu(X) \times \nu(Y)$.*

We will denote the number of 1's in a string $x \in \{0, 1\}^n$ by $|x|$.

**Definition 11** (Hamming distance). *The (Hamming) distance between any two strings $x, y \in \{0, 1\}^n$, denoted $d(x, y)$, is defined as $d(x, y) \triangleq |\{i : x_i \neq y_i\}|$. The Hamming distance between any two sets $A, B \subseteq \{0, 1\}^n$, denoted $d(A, B)$, is the minimum pairwise distance; $d(A, B) = \min_{x \in A, y \in B} d(x, y)$.*

**Definition 12** (Hamming balls). *Let $c \in \{0, 1\}^n$ and $k \in \{1, \ldots, n\}$. A set $A \subseteq \{0, 1\}^n$ is called a Hamming ball with centre $c$ and radius $k$ if*

$$\{s \in \{0, 1\}^n \mid d(s, c) \leq k - 1\} \subset A \subseteq \{s \in \{0, 1\}^n \mid d(s, c) \leq k\}.$$

*A singleton set $A = \{c\}$ is a Hamming ball with centre $c$ and radius 0.*

For a set $A \subseteq \{0,1\}^n$, the boundary of $A$ is the set $\{s \in \{0,1\}^n \mid d(s,A) = 1\}$. In [12], Harper proved a isoperimetry result: among all sets of a given size, Hamming balls have the smallest boundary set size. A simplified proof was given by Frankl and Füredi [7], who also stated the theorem in the equivalent form we mention below. (See also the presentation in [1]).

**Theorem 13** (Harper's Theorem). *Let $A, B \subseteq \{0,1\}^n$ be non-empty sets. Then, there exists a Hamming ball $A_0$ with centre $0^n$ and a Hamming ball $B_0$ with centre $1^n$ such that $|A_0| = |A|$, $|B_0| = B$, and $d(A_0, B_0) \geq d(A, B)$.*

**Definition 14** (Binary Entropy). *The binary entropy function $\mathbb{H} : [0,1] \to [0,1]$ is defined as follows: $\mathbb{H}(p) = -p \log p - (1-p) \log(1-p)$.*

**Fact 15.** $\mathbb{H}(1/4) < 0.82$.

# 3 Linear decision lists contain large monochromatic rectangles

The argument of Turán and Vatan from [18] implicitly showed that any function $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ with no large monochromatic squares cannot be computed by small linear decision lists. Their argument was presented specific to the Inner Product function (Theorem 1 in [18]). However, it is not too hard to see that their proof in fact works for any function as long as it has no large monochromatic squares. In this section, we generalize their argument to show that all functions computable by small size linear decision lists must contain, under *any product distribution*, a monochromatic rectangle of large weight with respect to that distribution.

We first establish a technical lemma that can be seen as a generalization of Lemma 2 in [18].

**Lemma 16.** *Let $f$ be an LTF over the input variables $x_1, \ldots, x_n, y_1, \ldots, y_n$. Let $\mu, \nu$ be distributions over $\{0,1\}^n$, and $X, Y \subseteq \{0,1\}^n$. Define $m := \min\{\mu(X), \nu(Y)\}$, and let $t \in (0, m]$. Then, one of the following is true.*

1. *There exists a monochromatic 1-rectangle $(X', Y')$ within $X \times Y$ (i.e., $X' \subseteq X$ and $Y' \subseteq Y$) such that $\mu(X') \geq t$ and $\nu(Y') \geq t$.*

2. *There exists a monochromatic 0-rectangle $(X', Y')$ within $X \times Y$ such that $\mu(X') > m - t$ and $\nu(Y') > m - t$.*

*Proof.* Let $M$ be the submatrix of $M_f$ restricted to $X \times Y$. Let the LTF $f$ be given by $\mathsf{sign}(a + \langle \alpha \cdot x \rangle + \langle \beta \cdot y \rangle)$. Reorder the rows and columns of $M$ in

decreasing order of $a + \langle \alpha \cdot x \rangle$ and $\langle \beta \cdot y \rangle$ to get the matrix $B = R \times C$. Let $i$ denote the least index of a row in $B$ such that $\mu(\{R_1, \dots R_i\}) \geq t$, and $j$ denote the least index of a column in $B$ such that $\mu(\{C_1, \dots C_j\}) \geq t$. Note that these indices are well-defined since $t \in (0, m]$. If the $[i, j]$'th entry of $B$ is 1, then the top-left submatrix of $B$ satisfies item (1) in the lemma. If the $[i, j]$'th entry of $B$ is 0, then the bottom-right submatrix of $B$ satisfies item (2) in the lemma. $\qquad \square$

We now prove the main lemma.

**Lemma 17.** *Let $\mu, \nu$ be distributions on $\{0, 1\}^n$. Let $f : \{0, 1\}^n \times \{0, 1\}^n \to \{0, 1\}$ be any function with no monochromatic rectangle of weight greater than $w$ under the distribution $\mu \times \nu$. Then, any linear decision list computing $f$ must have size at least $1/\sqrt{w}$.*

*Proof.* Towards a contradiction, let $(L_1, a_1), (L_2, a_2), \dots, (L_k, a_k)$ be an LDL of size $k$ computing $f$, where $k < 1/\sqrt{w}$. Pick any $t \in (\sqrt{w}, 1/k]$. We construct, for each $i \in [k-1]$, a rectangle $S_i = X_i \times Y_i$ which is a 0-rectangle for all $L_j$ with $j \leq i$, and furthermore $\mu(X_i), \nu(Y_i) \geq 1 - i \cdot t$. We proceed by induction on $i$.

For the base case $i = 1$, let $S_0 = (X_0, Y_0)$ be the entire $2^n \times 2^n$ matrix. Suppose $S_0$ has a rectangle $(X', Y')$ that is a 1-rectangle of $L_1$ and moreover, $\mu(X') \geq t$, $\nu(Y') \geq t$. Then everywhere in this rectangle, $f$ will be $a_1$. But $f$ has no monochromatic rectangle of weight as large as $t^2 > w$. So $S_0$ has no rectangle $(X', Y')$ with $\mu(X') \geq t$, $\nu(Y') \geq t$ that is a 1-rectangle of $L_1$. By Lemma 16, $S_0$ must then contain a 0-rectangle $(X_1, Y_1)$ of $L_1$ such that both $\mu(X_1)$ and $\nu(Y_1)$ are at least $1 - t$. This establishes the base case.

For the inductive step, we have a rectangle $S_{i-1} = (X_{i-1}, Y_{i-1})$ which is a 0-rectangle for $L_1, L_2, \dots, L_{i-1}$ and, moreover, $\min\{\mu(X_{i-1}), \nu(Y_{i-1})\} \geq 1 - (i-1)t$. Within this rectangle, suppose $L_i$ has a 1-rectangle $(X', Y')$ such that $\mu(X') \geq t$ and $\nu(Y') \geq t$. Then $f = a_i$ in this rectangle, giving a monochromatic rectangle of $f$ of weight greater than $w$. But we know that such rectangles do not exist. Since $kt \leq 1$ and $i < k$, we have $t \leq 1 - (i-1)t$ and hence Lemma 16 is applicable. Hence we conclude that $S_{i-1}$ must contain a 0-rectangle $(X_i, Y_i)$ of $L_i$ with $\min\{\mu(X_i), \nu(Y_i)\} \geq 1 - (i-1)t - t = 1 - it$. Since this rectangle, say $S_i$, is contained in $S_{i-1}$, it is a 0-rectangle for all $L_j$ with $j \leq i$.

Thus, we have a rectangle $S_{k-1} = (X_{k-1}, Y_{k-1})$ on which $L_1, L_2, \dots, L_{k-1}$ are 0, and $L_k = 1$ because $L_k$ is the constant function 1. Furthermore, $\mu(X_{k-1})$ and $\nu(Y_{k-1}) \geq 1 - (k-1)t$. Everywhere on this rectangle, $f$ evaluates to $a_k$. So $S_{k-1}$ is a monochromatic rectangle for $f$. Hence it cannot

have weight more than $w$. Thus $1 - (k-1)t \leq \sqrt{w} < t$; that is, $1 < kt$, contradicting our choice of $t$. $\qquad\square$

# 4  MAJ∘XOR has no large monochromatic squares

In this section, we show an upper bound and a matching tight lower bound on the size of a largest monochromatic square in the communication matrix of the $\mathsf{MAJ} \circ \mathsf{XOR}$ function.

**Definition 18** (XOR functions)**.** *For a function* $f : \{0,1\}^n \to \{0,1\}$, *let* $f \circ \mathsf{XOR}$ *denote the function defined by* $f \circ \mathsf{XOR}(x_1, \ldots, x_n, y_1, \ldots y_n) = f(x_1 \oplus y_1, \ldots, x_n \oplus y_n)$.

**Lemma 19.** *Let* $F : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ *be the function* $\mathsf{MAJ}_n \circ \mathsf{XOR}$. *Then, for any* $b \in \{0,1\}$, $M_F$ *has a monochromatic* $b$-*square of size at least* $\sum_{i=0}^{\lfloor n/4 \rfloor} \binom{n}{i}$.

*Proof.* Define the sets $X, Y, Z$ as follows:

$$\begin{aligned} X = Y &= \{x \in \{0,1\}^n : |x| \leq \lfloor n/4 \rfloor\}. \\ Z &= \{x \in \{0,1\}^n : |x| \geq n - \lfloor n/4 \rfloor\}. \end{aligned}$$

Note that $F(x,y) = 0$ for all $x \in X, y \in Y$, and $F(x,z) = 1$ for all $x \in X$, $z \in Z$. Thus $(X,Y)$ and $(X,Z)$ are a monochromatic 0-square and 1-square, respectively, each of size $\sum_{i=0}^{\lfloor n/4 \rfloor} \binom{n}{i}$. $\qquad\square$

**Remark 20.** *We remark that when* $n \equiv 3 \pmod 4$ *the above construction can be improved if we consider monochromatic rectangles. That is, for any* $b \in \{0,1\}$, $M_F$ *has a monochromatic* $b$-*rectangle* $(X_1, X_2)$ *such that* $|X_1| = \sum_{i=0}^{\lceil n/4 \rceil} \binom{n}{i}$ *and* $|X_2| = \sum_{i=0}^{\lfloor n/4 \rfloor} \binom{n}{i}$. *Indeed, let* $X = \{x \in \{0,1\}^n : |x| \leq \lceil n/4 \rceil\}$, $Y = \{x \in \{0,1\}^n : |x| \leq \lfloor n/4 \rfloor\}$ *and* $Z = \{x \in \{0,1\}^n : |x| \geq n - \lfloor n/4 \rfloor\}$. *Then, it is easily seen that* $(X,Z)$ *(resp.,* $(X,Y)$) *is a monochromatic* 1-*rectangle (resp.,* 0-*rectangle) of the claimed size.*

We now show that this bound is tight.

**Theorem 21.** *Let* $F : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ *be the function* $\mathsf{MAJ}_n \circ \mathsf{XOR}$. *For any* $n$, $M_F$ *has no monochromatic squares of size greater than* $\sum_{i=0}^{\lceil n/4 \rceil} \binom{n}{i}$.

*Proof.* Suppose, to the contrary, that there are sets $A, B \subseteq \{0, 1\}^n$ such that $|A| = |B| > \sum_{i=0}^{\lceil n/4 \rceil} \binom{n}{i}$ and $A \times B$ is a monochromatic 1-square in $M_F$. By the definition of $F$, this implies $d(A, B) > \lfloor n/2 \rfloor$. By Theorem 13, there exist Hamming balls $A_0$ around $0^n$, and $B_0$ around $1^n$ such that $|A_0| = |A|, |B_0| = |B|$ and $d(A_0, B_0) \geq d(A, B)$. The size lower bound enforces that the radius of $A_0$ and $B_0$ must be greater than $\lceil n/4 \rceil$, and since they are centered on $0^n$ and $1^n$, it follows that $d(A_0, B_0) \leq \lfloor n/2 \rfloor$. But then $d(A, B)$ is also at most $\lfloor n/2 \rfloor$. Hence, there exist $x \in A, y \in B$ such that $d(x, y) \leq \lfloor n/2 \rfloor$, which means $F(x, y) = \mathsf{MAJ}_n \circ \mathsf{XOR}(x, y) = 0$, which contradicts our assumption. Therefore, any monochromatic 1-square in $M_F$ has size at most $\sum_{i=0}^{\lceil n/4 \rceil} \binom{n}{i}$.

A similar argument shows the same upper bound on the size of monochromatic 0-squares. $\qquad \square$

Now we can put things together to prove our main theorem.

*Proof of Theorem 1.* Let $s_n$ be the minimum size of an $\mathsf{LDL}$ computing $\mathsf{MAJ}_n \circ$ $\mathsf{XOR}$. Further let $\mu$ and $\nu$ be uniform distributions over $\{0, 1\}^n$. Then, by Lemma 17 and Theorem 21, for all $n$ sufficiently large,

$$
\begin{aligned}
s_n &\geq \frac{2^n}{\sum_{i=0}^{\lceil n/4 \rceil} \binom{n}{i}} & \\
&\geq \frac{2^n}{2^{n \cdot H(1/4)}} & \text{using Stirling's approximation} \\
&\geq 2^{0.18n}. & \text{using Fact 15}
\end{aligned}
$$

$\qquad \square$

# 5   $\mathsf{LDL}$s and the threshold circuit hierarchy

In this section, we see how the class of functions computable by polynomial sized $\mathsf{LDL}$s fits into the low depth threshold circuit hierarchy. The reader is referred to Razborov's survey [16] for a detailed exposition on the low depth threshold circuits hierarchy.

## 5.1   Definitions

**Definition 22** ($\mathsf{MAJ}$). *Define* $\mathsf{MAJ}$ *to be the class of all functions computable by polynomial sized* $\mathsf{MAJ}$ *gates. Each input to the* $\mathsf{MAJ}$ *gate may be a constant 0 or 1, or a variable* $x_i$, *or its negation* $\neg x_i$.

**Definition 23** (LTF). *Define* LTF *to be the class of all functions computable by* LTF *gates.*

**Definition 24** (LDL). *Define* LDL *to be the class of all functions computable by polynomial sized linear decision lists.*

**Definition 25** ($\widehat{\text{LDL}}$). *Define* $\widehat{\text{LDL}}$ *to be the class of all functions computable by polynomial sized linear decision lists where, furthermore, weights of the linear threshold queries are integers with values bounded by a polynomial in the number of variables.*

**Definition 26** (Depth-2 classes). *For classes of functions* $\mathcal{C}, \mathcal{D}$, *define* $\mathcal{C} \circ \mathcal{D}$ *to be the class of functions computable by polynomial-sized depth-2 circuits, where the top gate computes a function from the class* $\mathcal{C}$, *and the bottom layer contains gates computing functions in* $\mathcal{D}$.

**Definition 27** ($\widehat{\text{PT}_1}$). *The class* $\widehat{\text{PT}_1}$ *consists of all functions* $f : \{0,1\}^n \to \{0,1\}$ *which can be represented by polynomial sized* MAJ $\circ$ PARITY *circuits.*

**Definition 28** (PT$_1$). *The class* PT$_1$ *consists of all functions* $f : \{0,1\}^n \to \{0,1\}$ *which can be represented by polynomial sized* LTF $\circ$ PARITY *circuits.*

(These are precisely the classes of polynomial threshold functions [2]; it is more convenient for us here to use the equivalent formulation as depth-2 circuits.)

In order to define classes given by the spectral representation of functions, we first recall a few preliminaries from Boolean function analysis.

Consider the real vector space of functions from $\{0,1\}^n \to \mathbb{R}$, equipped with the following inner product.

$$\langle f, g \rangle = \frac{1}{2^n} \sum_{x \in \{0,1\}^n} f(x)g(x) = \mathbb{E}_{x \in \{0,1\}^n}[f(x)g(x)].$$

For each $S \subseteq [n]$, define $\chi_S : \{0,1\}^n \to \{-1,1\}$ by $\chi_S(x) = (-1)^{\sum_{i \in S} x_i}$. It is not hard to verify that $\{\chi_S : S \subseteq [n]\}$ forms an orthonormal basis for this vector space. Thus, every $f : \{0,1\}^n \to \mathbb{R}$ has a unique representation as $f = \sum_{S \subseteq [n]} \widehat{f}(S) \chi_S$, where

$$\widehat{f}(S) = \langle f, \chi_S \rangle = \mathbb{E}_{x \in \{0,1\}^n}[f(x)\chi_S(x)].$$

**Definition 29** (PL$_1$). *The class* PL$_1$ *consists of all functions* $f : \{0,1\}^n \to \{0,1\}$ *for which* $\sum_{S \subseteq [n]} |\widehat{f}(S)| \leq \text{poly}(n)$.
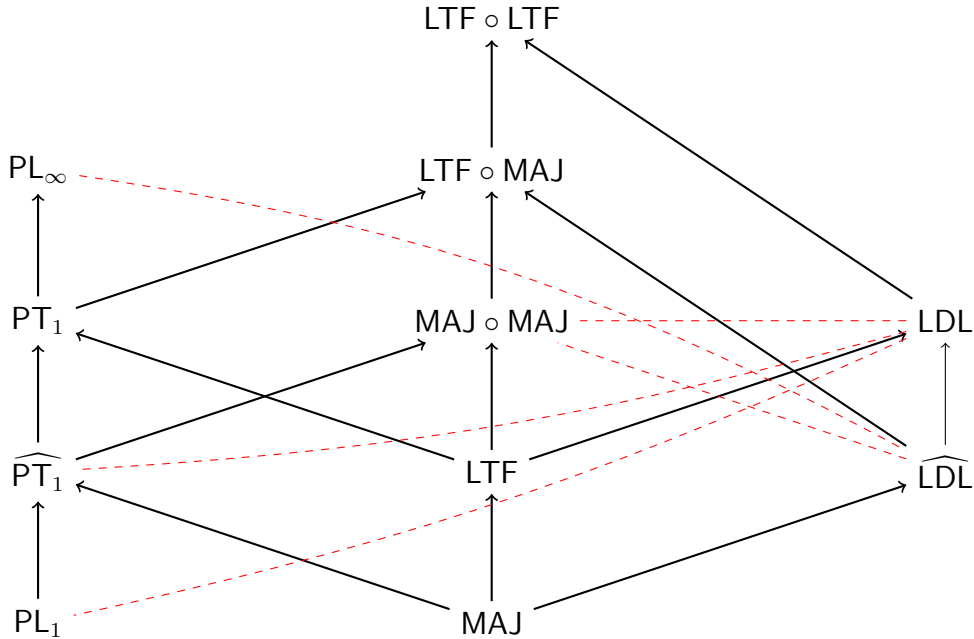
Figure 1: Low depth threshold circuit hierarchy

**Definition 30** ($\mathsf{PL}_\infty$). *The class $\mathsf{PL}_\infty$ consists of all functions $f : \{0,1\}^n \to \{0,1\}$ for which $\max\limits_{S \subseteq [n]} |\widehat{f}(S)| \geq \frac{1}{\mathrm{poly}(n)}$.*

Figure 1 depicts the currently known status of low depth circuit class containments, and shows where linear decision lists fit in this hierarchy.

A thick solid arrow from $\mathcal{C}_1$ to $\mathcal{C}_2$ denotes $\mathcal{C}_1 \subsetneq \mathcal{C}_2$, a thin solid arrow from $\mathcal{C}_1$ to $\mathcal{C}_2$ denotes $\mathcal{C}_1 \subseteq \mathcal{C}_2$, and a dashed line between $\mathcal{C}_1$ and $\mathcal{C}_2$ denotes incomparability. In the figure, we only show the newly established incomparabilities.

The leftmost column has the classes defined based on spectral representation, and the middle column has the classes based on depth-2 circuits. Concerning these classes, the picture was already completely clear: All containments shown among classes in these columns are known to be strict, and wherever no arrow connects two classes, they are known to be incomparable. Essentially this part of the figure appears in [8]; a subsequent refinement is the insertion of the class LTF ∘ MAJ, separated from MAJ ∘ MAJ in [8], from $\mathsf{PT}_1$ in [2] and most recently from LTF ∘ LTF in [4].

The two classes $\widehat{\mathsf{LDL}}$ and LDL form the new column on the right. In the following subsection we explain their position with respect to the other two

columns. However here the picture is not yet completely clear, and there are still several open questions.

## 5.2 New results

By definition, $\mathsf{MAJ} \subseteq \widehat{\mathsf{LDL}}$ and $\mathsf{LTF} \subseteq \mathsf{LDL}$ via lists of size 2. The parity function is known to not be in $\mathsf{LTF}$, and it has a simple $\mathsf{LDL}$ with 0-1 weights in the query functions. Thus both these containments are proper, and $\widehat{\mathsf{LDL}}$ is not contained in $\mathsf{LTF}$. We now observe that, implicit from prior work, $\widehat{\mathsf{LDL}}$ is not even contained in $\mathsf{MAJ} \circ \mathsf{MAJ}$.

**Theorem 31.**
$$\widehat{\mathsf{LDL}} \not\subseteq \mathsf{MAJ} \circ \mathsf{MAJ}.$$

*Proof.* Define the ODD-MAX-BIT function by $\mathsf{OMB}(x) = 1$ iff the largest index $i$ where $x_i = 1$ is odd ($\mathsf{OMB}(0^n) = 0$). Buhrman, Vereshchagin and de Wolf [3] showed that $\mathsf{OMB} \circ \mathsf{AND}$ is hard, in the sense that it has exponentially small *discrepancy*. By a result of Hajnal, Maass, Pudlák, Szegedy and Turán [10], this implies that $\mathsf{OMB} \circ \mathsf{AND}$ cannot be computed by polynomial sized $\mathsf{MAJ} \circ \mathsf{MAJ}$ circuits.

Note that $\mathsf{OMB}$ can be computed by a linear sized decision list by querying the variables in decreasing order of their indices. Thus $\mathsf{OMB} \circ \mathsf{AND}$ can be computed by a linear sized decision list of $\mathsf{AND}$'s, and hence by a linear decision list with 0-1 weights. $\square$

On the other hand, it is easily seen that $\mathsf{MAJ}_n \circ \mathsf{XOR}$ is in $\mathsf{MAJ} \circ \mathsf{MAJ}$, and even in $\widehat{\mathsf{PT}}_1$ (see for instance [2]). Combining this with Theorem 1, we obtain:

**Theorem 32.**
$$\widehat{\mathsf{PT}}_1 \not\subseteq \mathsf{LDL}.$$

The following strengthening of Theorem 32 is implicit from a recent result of Chattopadhyay, Mande and Sherif [5].

**Theorem 33.**
$$\mathsf{PL}_1 \not\subseteq \mathsf{LDL}.$$

(We defer a discussion of why Theorem 33 holds to Section 5.3.) Putting together these separations with the known containments $\mathsf{PL}_1 \subseteq \widehat{\mathsf{PT}}_1 \subseteq \mathsf{MAJ} \circ \mathsf{MAJ}$, we obtain a slew of incomparability results.

**Corollary 34.** *For any class $A \in \{\widehat{\mathsf{LDL}}, \mathsf{LDL}\}$ and $B \in \{\mathsf{PL}_1, \mathsf{MAJ} \circ \mathsf{MAJ}\}$, the classes $A$ and $B$ are incomparable.*

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In particular, the classes LDL and MAJ ∘ MAJ are incomparable. This completely answers the open question posed by Turán and Vatan [18].

Impagliazzo and Williams [13, Theorem 4.8] showed that the function $OR_n \circ EQ_n$ (also called Block-Equality) does not contain large monochromatic rectangles (in fact they showed that it does not contain large monochromatic rectangles under any product distribution). Thus, by Lemma 17, any linear decision list computing $OR_n \circ EQ_n$ must be of size at least $2^{\Omega(n)}$. We now observe that OR ∘ EQ ∈ MAJ ∘ MAJ. Consequently, OR ∘ EQ also witnesses MAJ ∘ MAJ $\not\subseteq$ LDL. However, in contrast to Theorem 1, note that the lower bound is subexponential since OR ∘ EQ is defined on $2n^2$ variables. Moreover, OR ∘ EQ seems to incur a significant polynomial blow up in size when simulated by MAJ ∘ MAJ circuits, whereas $MAJ_n \circ XOR$ has linear sized MAJ ∘ MAJ circuits.

**Theorem 35.**
$$OR \circ EQ \in MAJ \circ MAJ.$$

*Proof.* First observe that OR ∘ EQ can be computed by a MAJ ∘ EQ circuit by suitably padding constants to the input. Next, note that EQ is an *exact threshold function*, that is there exist reals $a_1, \ldots, a_n, b_1, \ldots, b_n, c$ such that $EQ(x, y) = 1$ iff $\sum_{i=1}^{n} a_i x_i + b_i y_i = c$. Hansen and Podolskii [11] showed that such functions can be efficiently simulated by MAJ∘LTF circuits. However, we do not need the full strength of their result, so we give a direct construction below.

For an equality on $2n$ bits, say $x_1, \ldots, x_n, y_1, \ldots, y_n$, note that

$$EQ_n(x_1, \ldots, x_n, y_1, \ldots, y_n) = 1 \iff \sum_{i=1}^{n} 2^i(x_i - y_i) = 0.$$

Consider the following linear threshold functions.

$$g_1(x, y) = \text{sign}\left(\sum_{i=1}^{n} 2^i(x_i - y_i) + 1/2\right) \text{ and}$$

$$g_2(x, y) = \text{sign}\left(\sum_{i=1}^{n} 2^i(x_i - y_i) - 1/2\right).$$

Observe that $g_1(x, y) - g_2(x, y) = EQ_n(x, y)$.

Let $g_1^{(i)}$ and $g_2^{(i)}$ denote these LTFs for the $i$th block on which we test equality. The function $OR_n \circ EQ_n$ is just

$$OR_n \circ EQ_n = \text{sign}\left((g_1^{(1)} - g_2^{(1)}) + (g_1^{(2)} - g_2^{(2)}) + \ldots + (g_1^{(n)} - g_2^{(n)})\right);$$

this formulation puts it in $\mathsf{MAJ} \circ \mathsf{LTF}$.

Finally, Goldmann, Håstad and Razborov [8] showed that $\mathsf{MAJ} \circ \mathsf{LTF} = \mathsf{MAJ} \circ \mathsf{MAJ}$. Thus, $\mathsf{OR} \circ \mathsf{EQ} \in \mathsf{MAJ} \circ \mathsf{MAJ}$. $\qquad\square$

**Theorem 36.**
$$\widehat{\mathsf{LDL}} \nsubseteq \mathsf{PL}_\infty.$$

*Proof.* It is easy to see that any symmetric function (a function that only depends on the Hamming weight of the input) can be computed by linear sized linear decision lists where query functions are majority: the linear threshold queries can be used to determine the Hamming weight of the input, and the decision list outputs the appropriate answer at each decision.

Bruck [2] showed that the *Complete Quadratic* function, which is a symmetric function, is not in $\mathsf{PL}_\infty$. This function yields the required separation. $\qquad\square$

Combining Corollary 34 and Theorem 36 yields more incomparability results.

**Corollary 37.** *For any class $A \in \{\widehat{\mathsf{LDL}}, \mathsf{LDL}\}$ and $B \in \{\mathsf{PL}_1, \mathsf{PL}_\infty\}$, the classes $A$ and $B$ are incomparable. In other words, all spectral classes in the first column (see Figure 1) are incomparable to all classes in the third column.*

Finally, as noted in [18], $\mathsf{LDL}$ is contained in $\mathsf{LTF} \circ \mathsf{LTF}$. The same argument shows that $\widehat{\mathsf{LDL}}$ is contained in $\mathsf{LTF} \circ \mathsf{MAJ}$. Corollary 34 implies that these containments are strict.

## 5.3  Proving Theorem 33

As mentioned earlier, it is implicit from a recent result of Chattopadhyay et al. [5] that $\mathsf{PL}_1 \nsubseteq \mathsf{LDL}$. We first define the function used to achieve the separation and introduce some background required.

**Definition 38** (SINK)**.** *Consider a complete undirected graph on $n$ vertices with variables $x_{i,j}$ for $i < j \in [n]$. The variable $x_{i,j}$ assigns a direction to the edge between $v_i$ and $v_j$ in the following way: $x_{i,j} = 0$ implies the edge points towards $v_i$, and $x_{i,j} = 1$ implies the edge points towards $v_j$. The function SINK computes whether or not there is a sink in the graph. In other words,*

$$\mathsf{SINK}(x) = 1 \iff \exists i \in [n] \text{ such that all edges adjacent to } i \text{ are incoming.}$$

We now define the notion of *projections* of strings to certain subsets of coordinates. Let $X \in \{0,1\}^{\binom{n}{2}}$. For any vertex $v_i$, let $E_{v_i}$ be the set of $n-1$ coordinates corresponding to the $n-1$ edges adjacent to $v_i$. Let $X_{v_i}$ denote the $(n-1)$-bit string obtained by projecting $X$ to the coordinates in $E_{v_i}$.

**Definition 39** (Entropy). *Let $X$ be a discrete random variable. The entropy $H(X)$ is defined as*

$$H(X) = \sum_{s \in \mathrm{supp(X)}} \Pr[X = s] \log \frac{1}{\Pr[X = s]}.$$

**Fact 40** (Folklore). $\mathrm{supp}(X) = k \implies H(X) \leq \log k$, *with equality if and only if $X$ is uniform.*

**Lemma 41** (Shearer's Lemma [6] (see also [15])). *Let $X = (X_1, \ldots, X_t)$ be a random variable. If $S$ is a set of projections such that for each $i \in [t]$, $i$ appears in at least $k$ projections, then $\sum_{P \in S}[H_{X_P}] \geq kH(X)$.*

Chattopadhyay et al. [5] introduced and used the function $\mathsf{SINK} \circ \mathsf{XOR}$ to refute the long-standing Log-Approximate-Rank Conjecture, along with several other conjectures. They observe that $\mathsf{SINK} \circ \mathsf{XOR} \in \mathsf{PL}_1$ [5, Theorem 1.10].

**Lemma 42** (Part 1 of Theorem 1.10 in [5]).

$$\mathsf{SINK} \circ \mathsf{XOR} \in \mathsf{PL}_1.$$

It is also implicit from their work that $\mathsf{SINK} \circ \mathsf{XOR}$ does not contain large monochromatic rectangles under the uniform distribution. More precisely, plugging the value $\epsilon = 0$ in [5, Claim 6.4] implies that any monochromatic rectangle in the communication matrix of $\mathsf{SINK} \circ \mathsf{XOR}$ on $2\binom{n}{2}$ variables must have weight at most $2^{2\binom{n}{2}-\Omega(n)}$. However, we do not require the full power of their proof for our purpose, and therefore produce a self-contained proof below.

**Theorem 43.** *Any monochromatic rectangle $R = A \times B$ in the communication matrix of $\mathsf{SINK} \circ \mathsf{XOR}$ must satisfy $|R| \leq 2^{2\binom{n}{2}-n+\log n+1}$.*

*Proof.* It is easy to verify that the probability of a 1-input under the uniform distribution equals $n/2^{n-1}$. Hence if $R$ is a 1-monochromatic rectangle, then $|R| \leq 2^{2\binom{n}{2}} \times n/2^{n-1}$, as claimed in the theorem.

Let $R = A \times B$ be a 0-monochromatic rectangle. Consider the random variable $XY$ ($X$ concatenated with $Y$) over $2\binom{n}{2}$ coordinates, when $X$ and

$Y$ are sampled uniformly from $A$ and $B$, respectively. From Fact 40 we have $H(XY) = \log |R|$.

Let $S$ be the set of projections $S := \{E_{v_i} \mid 1 \le i \le n\}$. Then each coordinate appears in exactly two projections. Hence by Lemma 41,

$$2H(XY) \le \sum_{P \in S} H((XY)_P) = \sum_{i \in [n]} H((XY)_{v_i}).$$

We now bound the entropy in $XY$ restricted to each of the projections. Let $A_{v_i}$ and $B_{v_i}$ be the projections of $A$ and $B$ on $E_{v_i}$, respectively. Since there is no input in $R$ which is a sink, we have $|\text{supp}(A_{v_i})| + |\text{supp}(B_{v_i})| \le 2^{n-1}$. (Each string in $A_{v_i}$ rules out one string from $B_{v_i}$ and vice versa.) By the AM-GM inequality, $|\text{supp}(A_{v_i})| \cdot |\text{supp}(B_{v_i})| \le 2^{2n-4}$. Hence Fact 40 implies that $H((XY)_{v_i}) \le 2n - 4$.

Returning to our use of Lemma 41, we obtain

$$2H(XY) \le \sum_{P \in S} H((XY)_P) \le n(2n-4)$$

$$\implies H(XY) \le 2\binom{n}{2} - n$$

$$\implies |R| \le 2^{2\binom{n}{2}-n}.$$

$\square$

Along with Lemma 17, Theorem 43 shows that any linear decision list computing the function $\mathsf{SINK} \circ \mathsf{XOR}$ on $2\binom{n}{2}$ variables (which is in $\mathsf{PL}_1$) must have size at least $2^{n/2}$. This completes the proof of Theorem 33.

Clearly, $\mathsf{SINK} \circ \mathsf{XOR}$ also witnesses $\mathsf{MAJ} \circ \mathsf{MAJ} \not\subseteq \mathsf{LDL}$. However, the lower bound against $\mathsf{LDL}$ is again only subexponential.

# 6 Conclusions

We show that $\mathsf{MAJ} \circ \mathsf{XOR}$ cannot be computed by polynomial sized linear decision lists, resolving an open question of Turán and Vatan [18]. We also show that several spectral classes and polynomial threshold function classes are incomparable to linear decision lists. Figure 1 depicts where the class $\mathsf{LDL}$, and its small-weight version $\widehat{\mathsf{LDL}}$, fit in the low depth threshold circuit hierarchy.

A subset of the authors [4] showed that a decision list of *exact threshold functions* cannot be computed by $\mathsf{LTF} \circ \mathsf{MAJ}$. A natural question that arises is whether $\mathsf{LDL}$ is incomparable with $\mathsf{LTF} \circ \mathsf{MAJ}$. (Note that the function

from [4] separating LTF ∘ LTF from LTF ∘ MAJ does not settle this question as it is also not in LDL – it contains the function OR ∘ EQ as a subfunction.)

Another natural question is whether $\widehat{\mathsf{LDL}}$ is strictly contained in LDL; that is, whether weights matter in linear decision lists.

# 7    Acknowledgments

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