# Plenitude

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**Abstract**

Oliver and Smiley's mid-plural logic is extended with indefinite descriptions.

## 1 Introduction

Plural logic has become popular, from the extensive literature we cite [Lin83, Boo84, Bur04, Yi05, FL17], for a recent criticism see [FN20]. Oliver and Smiley wrote a book on the subject [OS13] which we take for granted as our base and will frequently refer to. Here is a brief summary of what we require.

The basic idea in their extension of first-order logic to a two-sorted free logic is that variables are divided into singular (apple) and plural (**apples**) sorts. Combinations **a,b**, **a.b** and **-a** exhibit Boolean structure of plurals. The idiosyncratic notation reflects Oliver and Smiley's desire to avoid ontological commitment to sets. Plural variables do not come under the scope of quantification in *mid-plural* logic [OS13, Chapter 12], it is shown that there is an expressiveness jump both for plural definite descriptions (**the apples**) and plural quantifiers ($\forall$**apples**) rendering axiomatization impossible. By restricting quantification, Oliver and Smiley obtain completeness for mid-plural logic following a carefully worked out Henkin argument.

Our approach is as follows. When talking of existence, one can say there is only a single $x$ such that $A(x)$, or there exist exactly two $x \neq y$ such that $A(x)$ and $A(y)$, or there exist exactly three distinct $x, y, z$ such that $A(x)$, $A(y)$ and $A(z)$, and so on upto some finite number. Then we start running out of variables to talk. So we suggest $Many$ **a**, where **a** is of plural sort, to describe a large but finite number of singular variables ranging over the elements of **a**.

$Many$ and $Few$ are indefinite descriptions, they do not stand for a fixed quantity. In a domestic application one may say there are many apples in the refrigerator, when the number may be something like a dozen. In a societal application one may say there were many people in the market, when the number may be something like a thousand. Even in a single discourse, for the statements that there are many apples in the refrigerator and there were many people in the market, the magnitude of the two plurals may not be comparable.

There exist $Many$(**a,b**), that is, there are many individuals which are among **a** or **b**, does not imply that $Many$ **a** or that $Many$ **b**. This is discussed below. $Many$ **a** $\wedge$ $Many$ **-a** could be consistent, for example in a close election. $\neg Many$ **a** $\wedge \neg Few$ **a** could be consistent, we do not insist on having to decide among them. The range inbetween is called a *penumbra* [Fin75], following an idea from [PPP01] a predicate $Bet_{Few,Many}$ **a** could be introduced.

$Few$ **a** $\wedge$ $Few$ **-a** could be consistent because the domain may have only few elements. But this reduces to zeroth-order logic where quantifiers are not required. So for simplicity we assume the *plenitude* of the domain, that there are many values in the domain.

We follow Philip Peterson's work [Pet79] which had $Few\ x$, $Many\ x$ and $Most\ x$ quantifiers and bound variables ranging over individuals. In his logic $Many\ x\ A \supset Few\ x\ A$ is valid and $Few\ x\ A \equiv Most\ x\ \neg A$. We accept that $Few$ is a *degree* of plenitude lower than $Many$ [Urq86].

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Harvey Friedman interpreted there exists a set with many elements as a *generalized quantifier*: there exists a set of positive measure, and proved a completeness theorem [Fri78, Fri79, Ste85]. The treatment here is not quantified but *propositional*, linguistically justified by an interest in *finite* plurals. This article addresses Open Problem 2.7 of Moss and Raty [MR18].

## 2 Syntax

The signature of our logic contains all the symbols in mid-plural logic (which extends first-order logic) plus the indefinite plural descriptions. Thus there are singular variables for which we use $x, y, \ldots$, plural variables for which we use $\mathbf{a}, \mathbf{b}, \ldots$. There are also constants, function and predicate symbols forming terms of mixed sort for which we use $a, b, \ldots$, exhaustive and indefinite descriptions, combinations of plurals, connectives $\wedge, \vee, \neg, \supset, \equiv$ and quantifiers $\forall, \exists$.

Terms are as in Oliver and Smiley. They include variables and constants, function terms $f(a_1, a_2, ..., a_n)$, exhaustive descriptions $x{:}A$—which are not written $\{x{:}A\}$ to refrain from ontological commitment to sets—and plural combinations **-a**, **a.b**, **a-b** and **a,b** for complement, intersection, difference and union.

Atomic formulas include predicates $P(a_1, a_2, ..., a_n)$, inclusion $a \preccurlyeq b$, read as *a among b*, and indefinite descriptions $Many$ **a**, $Few$ **a**, $Most$ **a** (the last definable as $Few$ **-a**) and, if required, $Bet_{Few,Many}$ **a**.[1] A finite number of plenitude degrees (say $few, many, plenty$), possibly with penumbral predicates, can be allowed with small changes to the axioms and proofs below.

Atomic formulas equality $a = b$ and constants $true$ and $false$ are definable as two-way inclusion, $x = x$, etc. Boolean formulas are as usual. Exhaustive and indefinite descriptions can define quantifiers, $\exists x A$ as $x{:}A = x{:}A$, $\forall x A$ as $x{:}A = x{:}true$, $Many\ x\ A$ as $Many(x{:}A)$.

$A^\star$ distributes predicate $A$ over a plural, that is, $A^\star(\mathbf{a})$ when every $x \preccurlyeq \mathbf{a}$ satisfies $A(x)$. Using exhaustive description, "There are many apples in the refrigerator" can be written as $Many(x{:}\ apple(x) \wedge inFridge(x))$. By defining plurals such as **apples** $= x{:}apple(x)$, this can be written as $Many$(**apples.inFridge**). "Many apples are red" is $Many(\mathbf{apples}.red^\star)$.

Following Oliver and Smiley, a model has multi-valued interpretation of plural terms. Our interpretation is richer because it has to be said whether those multi-values are many, are few, so the interpretation now has degrees "many" or "few" or inbetween. A plural term **a.b** has degree many *only if* both constituents also do. A plural term **a,b** has degree many *if* either constituent does. The lack of the necessity/sufficiency of the conditions makes plurals indefinite.

## 3 Examples

One might say that there are few continents because one can count them on one's fingers. We want to avoid explicit information on counts. Extending this information to plural terms is not inferential. 8 planets is not many, whereas it appears that 80 planets would be many, a ten-fold division of many instances may result in each quotient having not many instances. For instance one may accept the truth of $Many$ **apples**, $Many$ **inOrchard** and $Many$ **inFridge** and of $Many$ **apples.inOrchard**, but not of $Many$ **apples.inFridge**. We accept that $Many$ **a.b** implies $Many$ **a** and $Many$ **b**.

The *pigeonhole principle* $Many$ **a,b** $\supset$ ($Many$ **a** $\vee$ $Many$ **b**) is not deemed valid. (Friedman had a corresponding axiom, see [Ste85].) This rules out a proof of a König lemma of the form: If in a tree with many nodes, all nodes have few chldren, then there must be a path in the tree with many nodes. This is because, beginning with a root with many descendants, an induction

[1] In supervaluations $Few$ **a** $\vee$ $Many$ **a** [Fin75], or in subvaluations $Few$ **a** $\wedge$ $Many$ **a** [Hyd97], could serve.

step would come to the few children of a node having many descendants. This is a collective and *non-distributive* property of these children [Yi99], it may be that each of 8 (few) children has 8 (few) descendants but together they have 64 (many). Multiplication of few by few could yield a collective many, see Gabbay and Schlechta in the context of non-monotonic logic [GS09].

To switch to something non-mathematical, there is a lively controversy among astronomers and the general public on whether there are many planets, hinging on what defines a planet. Thus, whether there is a planet which goes around the Sun once in 10000 years depends on the belief that there are many planets, as follows.

The belief that there are many planets is based on the principle that planethood is primarily determined by size. Other astronomical details such as rigidity and not being a satellite enter the picture, they are ignored here. Bodies with diameter more than 900 km and not "clearing" their orbits are called dwarf planets. Ceres and Pluto are dwarf planets. The orbit of Pluto intersects that of the traditional planet Neptune, but by taking Pluto to be a dwarf planet, Neptune's orbit is clearing. It is believed that there are more dwarf planets to be found, since they are smaller and difficult to observe.

Let the older planets in our solar system (except Pluto) come under the plural

$$\mathbf{clearing} = x\mathbf{:}\ \mathit{diam}(x) > 900 \wedge \mathit{clear}(x),$$

stating that their size is above 900 km, and they "clear" their orbit of other planets. The new category of dwarf planets is the plural

$$\mathbf{dwarf} = x\mathbf{:}\ \mathit{diam}(x) > 900 \wedge \neg \mathit{clear}(x).$$

The state of our solar system is the formula (most planets orbit the Sun in $< 300$ Earth years):

$$\begin{aligned} A = \quad & \mathit{Many}\ \mathbf{dwarf} \wedge \mathit{Few}\ \mathbf{clearing} \wedge \forall x(x \preccurlyeq \mathbf{clearing} \supset \mathit{orbit}(x) < 300) \wedge \\ & \mathit{pluto} \preccurlyeq \mathbf{dwarf} \wedge \mathit{ceres} \preccurlyeq \mathbf{dwarf} \wedge \mathit{orbit}(\mathit{pluto}) < 300 \wedge \mathit{orbit}(\mathit{ceres}) < 300. \end{aligned}$$

There are two definitions of planets, $\mathbf{planet_1} = \mathbf{clearing}$ then implies $\mathit{Few}\ \mathbf{planet_1}$, and $\mathbf{planet_2} = \mathbf{dwarf{,}clearing}$ implies $\mathit{Many}\ \mathbf{planet_2}$. Astronomers are divided on whether to include the dwarf planets among the planets (see [Bro10] for a popular account):

$$B = (\mathbf{planet} = \mathbf{planet_1}) \vee (\mathbf{planet} = \mathbf{planet_2}).$$

Sedna, discovered on 14 November 2003, is about the size of Ceres, although it is not yet classified as a dwarf planet because we do not know about its rigidity. Here is what we know.

$$C = (\mathit{diam}(\mathit{ceres}) > 900 \equiv \mathit{diam}(\mathit{sedna}) > 900) \wedge \mathit{orbit}(\mathit{sedna}) > 10000.$$

Therefore $A, C \vdash \mathit{sedna} \preccurlyeq \mathbf{planet_2}$. Since $\mathit{Many}\ \mathbf{planet}$ and $\mathit{Few}\ \mathbf{planet}$ cannot both be true (this is formalized as an axiom below), we get $A, B, C, \mathit{Many}\ \mathbf{planet} \vdash \mathit{sedna} \preccurlyeq \mathbf{planet}$.

By first-order logic, $A, B, C, \mathit{Many}\ \mathbf{planet} \vdash \exists x(x \preccurlyeq \mathbf{planet} \wedge \mathit{orbit}(x) > 10000)$.

# 4 Axioms

In addition to the axioms and inference rules in Oliver and Smiley, we add the following. Our semantics will rely on an intuitive notion of degrees of plenitude.

1. $\mathit{Many}\ (x\mathbf{:}\mathit{true})$ asserts there are many elements in the domain. This is an assumption.

2. $\forall x(x \preccurlyeq \mathbf{a} \supset x = \mathbf{a}) \supset Few\ \mathbf{a}$ says that singletons are few, so $few$ will end up getting a lower degree than any other.

3. $\mathbf{a} \preccurlyeq \mathbf{b} \supset (Many\ \mathbf{a} \supset Many\ \mathbf{b}) \wedge (Few\ \mathbf{b} \supset Few\ \mathbf{a})$ expresses monotonicity of plenitude. It follows that $Many\ \mathbf{a.b} \supset Many\ \mathbf{a}$ and $Many\ \mathbf{a} \supset Many\ \mathbf{a,b}$, and under the same antecedent $\mathbf{a} \preccurlyeq \mathbf{b}$, that $Many\ \mathbf{\text{-}b} \supset Many\ \mathbf{\text{-}a}$.

4. $Few\ \mathbf{a} \supset \neg Many\ \mathbf{a}$ *partitions* the plural sort into degrees: $few$, *many* and the penumbral $Bet_{Few,Many}$ (which could be ruled out by another axiom). Plural terms are created by exhaustive description of a formula $A$, so implicitly with every formula there is a degree.

5. $\neg Many\ \mathbf{a} \supset Many\ \mathbf{\text{-}a}$ relates *negation* to plenitude. If there are not many individuals among $a$, there will be many in the complement given that there are many elements in the domain. This partitions $Many\ \mathbf{a}$ into three degrees based on the degrees of $\mathbf{\text{-}a}$: $Many\ \mathbf{a} \wedge Many\ \mathbf{\text{-}a}$ and the remaining two obtained from the previous axiom.

6. $\forall x((Few\ \mathbf{a} \supset Few\ \mathbf{a,}x \vee Bet_{Few,Many}\ \mathbf{a,}x) \wedge (Many\ \mathbf{a} \supset Many\ \mathbf{a\text{-}}x \vee Bet_{Few,Many}\ \mathbf{a\text{-}}x) \wedge (Bet_{Few,Many}\ \mathbf{a} \supset (Few\ \mathbf{a\text{-}}x \vee Bet_{Few,Many}\ \mathbf{a\text{-}}x) \wedge (Many\ \mathbf{a,}x \vee Bet_{Few,Many}\ \mathbf{a,}x)))$ sets up the degree ordering in the presence of a penumbral predicate.

# 5 Semantics

Recall the semantics given for $(x{:}A)$ in Oliver and Smiley. Let $val$ be an assignment. It is lifted to exhaustive descriptions using the clause:

$val(x{:}A)$ are the individuals $val'(x)$ for every $x$-variant $val'$ of $val$ such that $val' \models A$.

We add a Tarskian definition. The semantics is intuitive and the axioms try to capture that intuition. The third clause illustrates how penumbral predicates could be handled.

**Definition 5.1.** *$val \models Many\ \mathbf{a}$ if and only if there are many among $val(\mathbf{a})$.*

*$val \models Few\ \mathbf{a}$ if and only if there are few among $val(\mathbf{a})$.*

*$val \models Bet_{Few,Many}\ \mathbf{a}$ iff there are more than few and less than many among $val(\mathbf{a})$.*

**Theorem 5.2** (Completeness)**.** *The axiomatization above extending Oliver and Smiley's is weakly complete.*

*Proof.* We work with finite theories. The Henkin-style completeness proof for mid-plural logic in Oliver and Smiley is followed, only the differences introduced by our syntax are specified.

In the Lindenbaum construction of the model from a *finite* consistent set $\Delta$, when a distinct singular $h \preccurlyeq \mathbf{a}$ for Henkin constant $h$ is encountered in the enumeration as a possible addition to a finite consistent set $\Delta$, its consistency with respect to $\Delta$ has to be checked.

Suppose $Many\ \mathbf{a}$ in $\Delta$. Find maximal $\mathbf{b}$ under the $\preccurlyeq$ order such that $\Delta \vdash \mathbf{b} \preccurlyeq \mathbf{a} \wedge \neg Many\ \mathbf{b}$. Count distinct singular $h \preccurlyeq \mathbf{a}$, this must be above the count of distinct singular $h \preccurlyeq \mathbf{b}$. If required create many fresh singular witnesses $h$ and add $h \preccurlyeq \mathbf{a}, h \preccurlyeq \mathbf{\text{-}b}, \exists x(x = h)$ to $\Delta$ (sufficient to cross intermediate degrees and penumbras), and also add the *plenitude conditions* $h \neq h'$ for every two such $h, h'$. By monotonicity and negation $Many(\mathbf{a,}h)$ is consistent when $Many\ \mathbf{a}$ is.

Suppose $Few\ \mathbf{a}$ in $\Delta$. Find minimal $\mathbf{b}$ under the $\preccurlyeq$ order such that $\Delta \vdash \mathbf{a} \preccurlyeq \mathbf{b} \wedge \neg Few\ \mathbf{b}$. Count distinct singular $h \preccurlyeq \mathbf{a}$, this must be below the count of distinct singular $h \preccurlyeq \mathbf{b}$ and adding one to the former should preserve the separation of counts.

Suppose a penumbral $Bet_{Few,Many}\ \mathbf{a}$ in $\Delta$. Find maximal $\mathbf{b}$ and minimal $\mathbf{c}$ under the $\preccurlyeq$ order such that $\Delta \vdash \mathbf{b} \preccurlyeq \mathbf{a} \wedge Few\ \mathbf{b}$ and $\Delta \vdash \mathbf{a} \preccurlyeq \mathbf{c} \wedge Many\ \mathbf{c}$ and preserve separation of counts.

Indefinite semantics of $Many$ and $Few$ are used here. What is required for completeness is that there are choices which lead to a maximal consistent set.

In the Truth Lemma for the model constructed in Oliver and Smiley [OS13, Chapter 12, Lemma 9], we have three additional cases to prove. Here is one of the requirements:

*The assignment* $val \models Many\ \mathbf{a}$ *if and only if the formula* $Many\ \mathbf{a}$ *is one of the truth set* $\Delta$.

Starting from the right, if $Many\ \mathbf{a}$ is one of the truth set $\Delta$, by our construction, many $h$ such that $h \preccurlyeq \mathbf{a}$ are in $\Delta$. By our plenitude conditions they are distinct. So many $x$-variants $val'$ of $val$ with $x \preccurlyeq \mathbf{a}$ exist. Since $\mathbf{a} = x{:}x \preccurlyeq \mathbf{a}$, there are many among $val(\mathbf{a})$ and $val \models Many\ \mathbf{a}$.

For the other direction, contrapositively, if $\neg Many\ \mathbf{a}$ is one of the truth set $\Delta$, our construction and the partition axiom ensure that at all stages there are not many $h$ such that $h \preccurlyeq \mathbf{a}$ were added to $\Delta$. Thus there are not many $x$-variants $val'$ of $val$ with $x \preccurlyeq \mathbf{a}$. Since $\mathbf{a} = x{:}x \preccurlyeq \mathbf{a}$, there are not many among $val(\mathbf{a})$ and $val \not\models Many\ \mathbf{a}$. □

*Remark* 5.3. Should one use a Skolem function for quantified formulas like $\forall x A(x, \mathbf{a})$ when the scope of the free plural $\mathbf{a}$ is within the quantifier? Thus in $\forall x(Many\ \mathbf{a} \wedge A(x, \mathbf{a}))$ where $\mathbf{a}$ appears only inside the quantifier, the plenitude of $\mathbf{a}$ will depend on the value of $x$.

# 6 Discussion

We thank Kit Fine, Anantha Padmanabha, Rohit Parikh, Abhisekh Sankaran, Byeong-Uk Yi and two anonymous referees for reading our earlier version and commenting on it.

Possible meanings for indefinite descriptions are through comparison classes [Sol11], intervals [Ret18] or (semi)lattices [Lin83, Bur16]. The structure of these scales, for example, the two scales of plenitude in $Many(\mathbf{apples.inFridge}) \wedge Many(\mathbf{apples.inOrchard})$, is not clear.

Indefinite descriptions can be applied to other areas, we take as our basis Kit Fine's Rutgers lectures on vagueness and sorites arguments [Fin20]. For vagueness there have been suggestions to use nonclassical logics [Dum75, Fin20], supervaluations [Fin75], games [Wri75, Par20], subvaluations [Hyd97] and bounded arithmetic [Par20]. Our approach extends that of degrees and intervals, anticipating the problem Fine mentions of "penumbral connection". Thus, in talking about two colours between which there is a penumbra, one could use a singular logic with predicates such as $Orange(x)$, $Red(x)$ and $Bet_{Orange,Red}(x)$ over an underlying linear order which may be discrete or dense.[2] Supervaluations which make more "precise" valuations correspond to shrinking the range of the penumbral predicate. In the most "precisified" view the penumbra shrinks to nothing. The approach of epistemic uncertainty is encompassed by leaving the boundaries between these various predicates indefinite.

Fine argues for global indeterminacy rather than local, that is, when doing a global "march" from orange to red, one is forced to choose a boundary. We rely on an irreflexive symmetric adjacency relationship between predicates [PPP01], allowing an indefinite $Bet_{Orange,Red}$ between adjacent colours, but not $Bet_{Yellow,Red}$ between distant colours (there could be applications for such predicates). The transitive order $\preccurlyeq$ in this article provides a more global context.

Parikh explains his development of Yessenin-Volpin's ultrafinitism [YV70] to show the utility of vague reasoning irrespective of semantics [Par20]. This article also seeks to increase expressiveness of the logical language disturbing the classical framework as little as possible. Whether there is cognitive access to plenitude [Yi18] is an interesting question.

[2] For example, the additive 3-byte RGB colour model in computer graphics has values $Orange = (255, 128, 0)$ and $Red = (255, 0, 0)$.

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